The index theorem for topological manifolds

by

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Dedicated to M. F. Atiyah and I. M. Singer

§ 0. Introduction

The famous Atiyah–Singer index theorem expresses the Fredholm index of any elliptic pseudo-differential operator on an arbitrary closed smooth manifold as the period of a universal algebraic combination of the Pontrjagin classes of the manifold, and of the Chern character of the symbol of the operator.

In 1969 M. F. Atiyah [A_1]—see also [A_2]—by taking as axioms some of the basic properties of the elliptic pseudo-differential operators of order zero on closed smooth manifolds, introduces the notion of abstract elliptic operator, for any compact topological space.

In 1970 I. M. Singer [Si] exposes a comprehensive program aimed to extend the theory of ellipic operators and their index to more general situations.

D. Sullivan's theorem [Su_1] about existence of an orientation class in the K-theory, in the world of odd primes, of PL-bundles over PL-manifolds, gives evidence that at least the *symbol* of *signature operators* on PL-manifolds should exist, see I. M. Singer [Si] § 3. One of the problems I. M. Singer formulated in [Si] was that of realizing, geometrically, signature operators on PL-manifolds, and at the same time, pointed out that such a realization could lead to an analytical proof of Novikov's theorem about the topological invariance of the rational Pontrjagin classes.

In 1975, L. G. Brown, R. G. Douglas, P. A. Fillmore [B.D.F.], and G. G. Kasparov [K] show, independently, in different contexts, that the stably-homotopic classes of abstract elliptic operators on the compact metric space X form an abelian group, which is naturally isomorphic to $K_0(X)$, the Spanier–Whitehead dual of $K^0(X)$. The same year P. Baum, W. Fulton, R. MacPherson [B.F.M.] prove the Riemann–Roch theorem for singular varieties.

In 1976, D. Sullivan discovers that any topological manifold of dimension $\neq 4$ admits a unique—up to homeomorphisms close to the identity—Lipschitz structure.

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