The Hausdorff dimension of the limit set of a geometrically finite Kleinian group

by

PEKKA TUKIA

University of Helsinki, Helsinki, Finland

A. Introduction

We consider in this paper groups of Möbius transformations of $\bar{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$. Such a group is called *Kleinian* if it acts discontinuously somewhere in $\bar{\mathbf{R}}^n$. The action of G extends to the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}=\mathbf{R}^n\times(0,\infty)$ and G is *geometrically finite* if there is a hyperbolic fundamental polyhedron with a finite number of faces for the action of G in $\mathbf{H}^{n+1}$ (for a more precise definition see [15, 1 B]). We prove in this paper that the *Hausdorff dimension* $\dim_{\mathrm{H}} L(G)$ of the limit set $L(G)$ of a geometrically finite Kleinian group G of $\bar{\mathbf{R}}^n$ is less than n (Theorem D).

Our proof of this theorem is based on the following observation. Assume that G is of *compact type*, i.e. if $\bar{\mathbf{H}}^{n+1}=\mathbf{H}^{n+1}\cup\bar{\mathbf{R}}^n$, then $(\bar{\mathbf{H}}^{n+1}\setminus L(G))/G$ is compact. Then there is an integer q such that if we divide any n-cube Q of $\mathbf{R}^n$ into q^n equal subcubes, then at least one of these subcubes does not touch $L(G)$. Let $\mathscr{L}(Q)$ be the family of these subcubes which touch $L(G)$. Then the n-measures of $Q'\in\mathscr{L}(Q)$ do not add up to the n-measure of Q and we get

$$\sum_{Q'\in\mathscr{L}(Q)} d(Q')^\alpha\leq c\,d(Q)^\alpha \tag{A 1}$$

for $\alpha=n$ and $c=1-1/q^n$. Obviously, this remains valid for slightly smaller $\alpha<n$ and slightly bigger $c<1$. Passing now to the families $\mathscr{L}(Q')$, $Q'\in\mathscr{L}(Q)$, we get an inductive argument which shows that the Hausdorff dimension of $L(G)$ cannot exceed $\alpha<n$.

The existence of such q is based on a compactness argument. If $r=d(Q\cap L(G))/d(Q)$ is small, the existence of such q is clear. On the other hand, if say $r\geq 1/2$, let z_Q be the center of Q and let s_Q be its side length. If $\bar{z}_Q=(z_Q, s_Q)\in\mathbf{H}^{n+1}$, the hyperbolic distance of $\bar{z}_Q$ from the hyperbolic convex hull H_G of $L(G)$ (see Section