Hausdorff dimension and capacities of intersections of sets in n-space

by

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1. Introduction

Suppose A and B are Borel sets in the Euclidean n space $\mathbf{R}^n$. If they are sufficiently nice, for example C^1 submanifolds or rectifiable of dimensions k and m with $k+m \geqslant n$, then according to a well-known formula of integral-geometry

$$\int \mathcal{H}^{k+m-n}(A \cap fB)\, d\lambda_n f = c(k, m, n)\, \mathcal{H}^k(A)\, \mathcal{H}^m(B),$$

where $\mathcal{H}^s$ stands for the s dimensional Hausdorff measure and λ_n is an invariant measure on the group of isometries of $\mathbf{R}^n$. Thus in this case there is a precise relation between the measures of A and B and of those of the intersections $A \cap fB$. The object of this paper is to study to what extent there are such relations, necessarily less precise, if either A or both A and B are completely general except for measurability assumptions. Thus various Cantor type sets, graphs of nowhere differentiable functions etc. should be included in our theory. Particular examples are the self-similar fractals, which Mandelbrot [MB] has considered in connection of several physical phenomena and for which Hutchinson [H] has presented a unified theory.

First to consider this problem was Marstrand [MJ] who explored the geometric properties of fractional dimensional subsets of the plane $\mathbf{R}^2$. He proved that if $A \subset \mathbf{R}^2$ is $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$, $1 < s < 2$, then for $\mathcal{H}^s$ almost all $x \in A$ $\dim A \cap l = s-1$ and $\mathcal{H}^{s-1}(A \cap l) < \infty$ for almost all lines l through x. He also showed by an example that $\mathcal{H}^{s-1}(A \cap l)$ may be zero for almost all lines l through any point of A. Marstrand's theorem was generalized to subsets of $\mathbf{R}^n$ with lines replaced by m planes in [MP1]. In [MP2] a potential-theoretic approach to this problem was presented. It was shown that