Every covering of a compact Riemann surface of genus greater than one carries a nontrivial L^2 harmonic differential

by

JOZEF DODZIUK [1]

Queens College, City University of New York
Flushing, NY, U.S.A.

This paper contains the proof of the assertion in its title. My motivation for considering this problem was the following. Let M be a compact differentiable manifold and let D be an elliptic differential operator between the spaces of C^∞ sections of two bundles on M. In [4] M. F. Atiyah proves that if index $D>0$, then, for every Galois covering $\tilde{M}\to M$, the operator $\tilde{D}$ induced by D on $\tilde{M}$ has nontrivial L^2 solutions $\tilde{D}u=0$. Atiyah goes on to ask whether the same is true for coverings which are not Galois. His guess is that the answer is negative in general but that the counter-example will be difficult to construct. The simplest situation to consider in this connection is that of a surface and the operator whose index is the Euler characteristic. For infinite coverings, since every L^2 harmonic function would be constant and nonzero constants are not in L^2, Atiyah's question reduces to the question of existence of L^2 harmonic forms of degree one. It is shown here that a counter-example will not be found in this simple setting.

From now on S will denote an oriented, compact surface of genus $g=g(S)>1$ equipped with a smooth Riemannian metric. $\tilde{S}\xrightarrow{\pi}S$ will be an arbitrary (usually infinite) covering of S, and $\Delta=\tilde{\Delta}$ will be the Laplace operator on $\tilde{S}$ with respect to the pull back metric. A differential (exterior form of degree one) on $\tilde{S}$ is harmonic, i.e. $\Delta\omega=0$, and in L^2 if and only if (cf. [15], Theorem 26)

$$d\omega = d*\omega = 0$$

and

$$\int_{\tilde{S}}|\omega|^2=\int_{\tilde{S}}\omega\wedge *\bar{\omega}<\infty.$$

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