A PROOF OF A CONJECTURE OF LOEWNER AND OF THE CONJECTURE OF CARATHEODORY ON UMBILIC POINTS

(Dedicated to the Memory of Charles Loewner)

BY

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1. Introduction

The Loewner Conjecture was motivated by the study of umbilic points on surfaces and by various other geometrical investigations concerning the qualitative theory of differential and integral operators (see especially Loewner [8]). Let u be a real analytic function on the disk D, $x^2+y^2<1$; with $2\partial_{\bar{z}}=\partial_x+i\partial_y$, think of the iterates $\partial_{\bar{z}}^n u$ of $\partial_{\bar{z}}$ on acting u as vector fields on D.

LOEWNER CONJECTURE (about 1950).

If the vector field $\partial_{\bar{z}}^n u$, $u \in C^\omega(D, \mathbf{R})$, $n \geqslant 1$, has an isolated zero at the origin, then the index of $\partial_{\bar{z}}^n u$ at the origin is not greater than n.

For $n=1$ this conjecture follows directly from standard techniques in differential equations (see Lefschetz [6]). For $n=2$ it is the key lemma required for a proof of the Caratheodory Conjecture (see Hamburger [3, 4], Bol [1], Klotz [5]).

CARATHEODORY CONJECTURE. *Every convex real analytic imbedding of S^2 in $\mathbf{E}^2$ has at least two umbilic points.*

With a proof of the Loewner Conjecture for $n=2$ the work of Hamburger together with standard more recent work in differential geometry will show that every real analytic immersion of S^2 in $\mathbf{E}^3$ has at least two umbilic points so that the convexity condition is in fact irrelevant.

The main difficulty in the proof of these conjectures occurs because the multiplicity of the zero of the vector field may be arbitrarily large; with standard conditions of genericity imposed the proofs of the Loewner Conjecture become relatively trivial. However, possibly