PHYSICAL STATES ON A C^*-ALGEBRA

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1. Introduction

Let A be a C^*-algebra with identity $\mathbf{1}$. A *physical state* is a function ϱ: $A \to \mathbf{C}$ which is a state on each singly generated C^*-subalgebra of A. Here "singly generated" means generated by $\mathbf{1}$ and a single self-adjoint element $a \in A$. The present paper is devoted to a discussion of whether a physical state ϱ on A is *linear*, i.e. whether it is a state in the ordinary sense. In the proper physical interpretation, this is the problem of linearity of the expectation functional on the algebra of observables in quantum mechanics, cf. Mackey [8] and Kadison [6].

Mathematically, the problem is also closely related to the following problem: Let R be a von Neumann algebra, and let P be the lattice of orthogonal projections in R. A function μ: $P \to \mathbf{R}^+$ such that $\mu(0) = 0$ is called a *completely additive measure* on P if

$$\mu(\sum_{i \in I} e_i) = \sum_{i \in I} \mu(e_i)$$

for any family $\{e_i\}_{i \in I}$ of mutually orthogonal projections in P. μ is a *probability measure* if $\mu(\mathbf{1}) = 1$. Given a probability measure μ on P one may ask whether there exists a positive normal state ϱ on R such that $\varrho | P = \mu$. This question, which poses what we may call the extension problem for measures (in non-commutative setting), was first suggested by Mackey. An affirmative solution for the special case where $R = \mathfrak{L}(H) =$ all bounded linear operators on a separable Hilbert space H, with $\dim H \geqslant 3$, was given in an ingenious paper by Gleason [5]. In the case where the measure is the dimension-function on the projections of a type II_1-factor, the problem of extension is precisely the problem of the additivity of the trace [7], [9].

The connection between the extension problem for measures and the linearity problem

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