GENERA AND DECOMPOSITIONS OF LATTICES OVER ORDERS

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Let k be an algebraic number field of finite degree, A/k a semi-simple finite-dimensional algebra over k and o a Dedekind ring with quotient field k. We consider o-orders R in A, that is, subrings with $kR = A$ and $1 \in R$, such that R is finitely generated as an o-module. An important example is the group ring oG of a finite group G, which is an order in kG. An R-lattice M is a finitely generated (unital) R-module, which is torsion-free as an o-module. The category of R-lattices we denote by $\mathfrak{L}_R$. For every prime ideal p in o, let o_p be the p-adic completion and put $R_p = o_p \otimes R$, $M_p = o_p \otimes M$ etc. Then R_p is an o_p-order in A_p and M_p is an R_p-lattice.

Two R_p-lattices M and N belong to the same genus—notation $M \sim N$—if $M_p \cong N_p$ as R_p-modules for every p. By $\mathcal{G}_R$ we denote the category of genera of R-lattices. It is well-known, that $M \sim N$ does not in general imply $M \cong N$; but the number of isomorphism classes in a genus is finite. In the present paper, we first give a classification of these isomorphism classes by means of ideal classes in the integral closure over o of the center of A (Theorem 2.2). This generalizes results of an earlier paper (Jacobinski [9]). The proof makes use of the classical theory of maximal orders and here we need the assumption, that k is an algebraic number field and o Dedekind. For a very small exceptional class of R-lattices, the classification is not complete. This is due to the fact, that maximal orders in a totally definite skew-field of index 2 have rather irregular properties.

We then use our result on the isomorphism classes in a genus to study various properties of R-lattices. As an immediate consequence we obtain an upper bound—depending only on R—for the number of isomorphism classes in a genus (Prop. 2.7).

An R-lattice X is called a local direct factor of M, if for every p, X_p is isomorphic to a direct factor of M_p. We show (Theorem 3.3), that then M has a decomposition of the form $M = X' \oplus N$, with $X' \sim X$. In a very special case—$R = oG$ the group ring of a finite