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On measure rigidity of unipotent subgroups of semisimple groups

by

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1. Introduction

This paper represents part II in our three part series on Raghunathan's measure conjecture (see [R4] for part I).

More specifically, let $\mathbf{G}$ be a real Lie group (all groups in this paper are assumed to be second countable), $\mathbf{\Gamma}$ a discrete subgroup of $\mathbf{G}$ and $\pi: \mathbf{G} \to \mathbf{\Gamma}\backslash\mathbf{G}$ the projection $\pi(\mathbf{g}) = \mathbf{\Gamma g}$. The group $\mathbf{G}$ acts by right translations on $\mathbf{\Gamma}\backslash\mathbf{G}$, $(x, \mathbf{g}) \to x\mathbf{g}, x \in \mathbf{\Gamma}\backslash\mathbf{G}, \mathbf{g} \in \mathbf{G}$. Let μ be a Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$. Define

$$(*) \qquad \mathbf{\Lambda}(\mu) = \mathbf{\Lambda}(\mathbf{G}, \mathbf{\Gamma}, \mu) = \{\mathbf{g} \in \mathbf{G}: \text{the action of } \mathbf{g} \text{ preserves } \mu\}.$$

The set $\mathbf{\Lambda}(\mu)$ is a closed subgroup of $\mathbf{G}$. The measure μ is called *algebraic* if there exists $\mathbf{x} = \mathbf{x}(\mu) \in \mathbf{G}$ such that $\mu(\pi(\mathbf{x})\, \mathbf{\Lambda}(\mu)) = 1$. In this case $\mathbf{x\Lambda}(\mu)\mathbf{x}^{-1} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{x\Lambda}(\mu)\, \mathbf{x}^{-1}$.

Definition 1. Let $\mathbf{U}$ be a subgroup of $\mathbf{G}$. We say that the action of $\mathbf{U}$ on $\mathbf{\Gamma}\backslash\mathbf{G}$ is *measure rigid* if every ergodic $\mathbf{U}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$ is algebraic. The group $\mathbf{U}$ is called *measure rigid* in $\mathbf{G}$ if its action on $\mathbf{\Gamma}\backslash\mathbf{G}$ is measure rigid for every lattice $\mathbf{\Gamma} \subset \mathbf{G}$. An element $\mathbf{u} \in \mathbf{G}$ is measure rigid if the group $\{\mathbf{u}^k: k \in \mathbf{Z}\}$ is measure rigid. $\mathbf{U} \subset \mathbf{G}$ and $\mathbf{u} \in \mathbf{G}$ are called *strictly measure rigid* if their action on $\mathbf{\Gamma}\backslash\mathbf{G}$ is measure rigid for *every* discrete subgroup $\mathbf{\Gamma}$ of $\mathbf{G}$.

A subgroup $\mathbf{U}$ of $\mathbf{G}$ is called unipotent if for each $\mathbf{u} \in \mathbf{U}$ the map $\mathrm{Ad}_{\mathbf{u}}$ is a unipotent automorphism of the Lie algebra of $\mathbf{G}$.

RAGHUNATHAN'S MEASURE CONJECTURE. *Every unipotent subgroup of a connected Lie group* $\mathbf{G}$ *is measure rigid.*

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