Acta Math., 165 (1990), 189–227

Algebraic L^2 decay for Navier–Stokes flows in exterior domains

by

WOLFGANG BORCHERS and TETSURO MIYAKAWA

Universität Paderborn
Paderborn, West Germany

Hiroshima University
Hiroshima, Japan

1. Introduction

In this paper we deduce algebraic decay rates for the total kinetic energy of weak solutions of nonstationary Navier–Stokes equations in exterior domains $\Omega\subset\mathbf{R}^n$, $n\geqslant 3$:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}+v\cdot\nabla v-\Delta v+\nabla p=0 \quad \text{in} \quad (0,\infty)\times\Omega \\ &\nabla\cdot v=0 \quad \text{in} \quad (0,\infty)\times\Omega \\ &v|_{\partial\Omega}=0; \quad v\to 0 \quad \text{as} \quad |x|\to\infty, \\ &v|_{t=0}=a. \end{aligned}\right\} \tag{NS}$$

Here $v=(v_1,\dots,v_n)$ and p denote, respectively, unknown velocity and pressure, while $a=(a_1,\dots,a_n)$ is a given initial velocity. By exterior domain we mean a connected open set Ω whose complement is the closure of the union of a finite number of bounded domains with C^∞ boundaries. For problem (NS) the existence of a weak solution in L^2 was first established by Hopf [16] for an arbitrary L^2-initial velocity. The uniqueness and the regularity of Hopf's weak solutions are still open questions.

The square of the L^2-norm of the fluid velocity v is proportional to the kinetic energy of the fluid under consideration; so in view of the presence of the viscosity term Δv and the no-slip boundary condition $v|_{\partial\Omega}=0$, it is reasonable to expect that the solution v would decay in L^2 as $t\to\infty$. However, it is in general not easy to deduce the expected L^2 decay property for the Navier–Stokes problem in unbounded domains. This L^2 decay problem was first raised by Leray [24] in the case of the Cauchy problem in $\mathbf{R}^3$ and then was affirmatively solved by Kato [20] for the Cauchy problem in $\mathbf{R}^3$ and $\mathbf{R}^4$ by using the fact that Leray's weak solutions become regular after a finite time.