THE RADON TRANSFORM ON EUCLIDEAN SPACES, COMPACT TWO-POINT HOMOGENEOUS SPACES AND GRASSMANN MANIFOLDS

BY

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§ 1. Introduction

As proved by Radon [16] and John [13], a differentiable function f of compact support on a Euclidean space $\mathbf{R}^n$ can be determined explicitly by means of its integrals over the hyperplanes in the space. Let $J(\omega, p)$ denote the integral of f over the hyperplane $\langle x, \omega\rangle = p$ where ω is a unit vector and $\langle,\rangle$ the inner product in $\mathbf{R}^n$. If Δ denotes the Laplacian on $\mathbf{R}^n$, $d\omega$ the area element on the unit sphere $\mathbf{S}^{n-1}$ then (John [14], p. 13)

$$f(x) = \tfrac{1}{2}(2\pi i)^{1-n}(\Delta_x)^{\frac{1}{2}(n-1)} \int_{\mathbf{S}^{n-1}} J(\omega, \langle \omega, x\rangle)\, d\omega, \quad (n \text{ odd}); \tag{1}$$

$$f(x) = (2\pi i)^{-n}(\Delta_x)^{\frac{1}{2}(n-2)} \int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega, p)}{p - \langle \omega, x\rangle}, \quad (n \text{ even}), \tag{2}$$

where, in the last formula, the Cauchy principal value is taken.

Considering now the simpler formula (1) we observe that it contains two dual integrations: the first over the set of points in a given hyperplane, the second over the set of hyperplanes passing through a given point. Generalizing this situation we consider the following setup:

(i) Let X be a manifold and G a transitive Lie transformation group of X. Let Ξ be a family of subsets of X permuted transitively by the action of G on X, whence Ξ acquires a G-invariant differentiable structure. Here Ξ will be called the *dual* space of X.

(ii) Given $x \in X$, let $\check{x}$ denote the set of $\xi \in \Xi$ passing through x. It is assumed that each ξ and each $\check{x}$ carry measures μ and ν, respectively, such that the action of G on X and Ξ permutes the measures μ and permutes the measures ν.

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