The Burgess Davis inequalities via Fefferman's inequality*

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Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ be a sequence of σ-fields such that $\mathcal{F} = \bigcup_{\nu=0}^{\infty} \mathcal{F}_\nu$.

For a random variable $f \in L_1(\Omega, \mathcal{F}, P)$ we shall set

$$f_n = E(f|\mathcal{F}_n), \quad \Delta f_n = f_n - f_{n-1},$$
$$f_n^* = \max_{\nu \le n} |f_\nu|, \quad f^* = \sup_n |f_n|,$$
$$S_n(f) = \sqrt{\textstyle\sum_{\nu=1}^n \Delta f_\nu^2}, \quad S(f) = \sup S_n(f).$$

We also introduce the spaces

$$\mathcal{X}_p = \{f\colon E([S(f)]^p) < \infty\} \tag{I.1}$$

with norm

$$\|f\|_{\mathcal{X}_p} = [E([S(f)]^p)]^{1/p} \quad (p \ge 1).$$

Furthermore, we let

$$\mathrm{BMO} = \{f\colon \sup_{n \ge 1} \|E(|f - f_{n-1}|^2|\mathcal{F}_n)\|_\infty < \infty\} \tag{I.2}$$

with norm**

$$\|f\|_{\mathrm{BMO}} = \sup_{n \ge 1} \|\sqrt{E(|f - f_{n-1}|^2|\mathcal{F}_n)}\|_\infty.$$

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* This work was supported by the Air Force Office of Scientific research under Grant AF-AFOSR 1322-67.

** Strictly speaking, these functionals are norms in the usual sense only when restricted to $\{f\colon E(f|\mathcal{F}_0) = 0\}$.