Degenerations of minimal ruled surfaces

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This paper studies degenerations of minimal ruled surfaces. With an additional assumption satisfied, after base change, by projective degenerations, we find that the minimal models of these degenerations are non-singular and particularly elementary: $\mathbf{P}^1$-bundles over minimal models of degenerations of the base curves.

Let $\pi: X \to \Delta$ be a proper flat holomorphic map from a non-singular complex threefold X onto Δ, the unit disk in $\mathbf{C}$, such that $X_t := \pi^{-1}(t)$ is a non-singular minimal ruled surface for $t \neq 0$ and X_0 is a union of non-singular surfaces meeting normally. In [11] p. 83, Persson finds an example of such a degeneration of minimal ruled surfaces for which X_0 is not algebraic, in fact, X_0 is a union of two Hopf surfaces, while the general fiber, X_t, is a minimal ruled elliptic surface. Of particular interest in this example is the fact that for S_t a section of X_t under its ruling, there is an analytic surface $S' \subset X \setminus X_0$ for which $S' \cap X_t = S_t$; however, S' does not extend to an analytic surface S over all of Δ.

In light of the above example, in this paper by *degeneration of minimal ruled surfaces,* we mean a map π as above for which $\pi = \pi_S \circ f$ where f: $X \to S$ is a holomorphic map, S is a non-singular complex surface, π_S: $S \to \Delta$ is a degeneration of curves for which $f|_{X_t}$: $X_t \to S_t$ gives X_t its structure of minimal ruled surface (see 1.1 for details). These degenerations are ruled degenerations, that is, X_t is being degenerated together with its structure as ruled surface. The central result of this paper is that, for X as above, X has a smooth minimal model which is a $\mathbf{P}^1$-fibration over S. The condition that such an S exists is both essential and quite natural: if $X \to \Delta$ is a degeneration of surfaces in the previous sense which is bimeromorphic over Δ to a projective degeneration, then, after a base-change and shrinking Δ, there exists an S as above (Persson [11], pp. 60 and 77). On the other hand, there are projective degenerations of minimal ruled surfaces for which no such S exist. For example, let Z be the blow-up of $\mathbf{P}^2$ at the 54 nodes of a nodal elliptic curve of degree 12. For C the resulting smooth elliptic curve, there is a projective conic bundle π: $X \to Z$ having C as its degeneration divisor and so X is not rational