One-sided minima of indefinite binary quadratic forms and one-sided diophantine approximations

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1. Introduction

Let

(1) $$f(x, y) = ax^2 + bxy + cy^2$$

be an indefinite binary quadratic form with real coefficients. We shall be concerned with the set of possible values of

(2) $$m_+(f) = \inf \frac{f(x, y)}{\sqrt{d}},$$

the infimum being taken over integers x, y for which $f(x, y) > 0$. Here $d = b^2 - 4ac$ is the discriminant of f. This is an analogue of the ordinary Markov spectrum, which is the set of possible values of $1/m(f)$, where $m(f) = \inf |f(x, y)|/\sqrt{d}$.

It is sometimes convenient to consider

(3) $$\lambda_+(f) = m_+(f)^{-2}.$$

We always have $\lambda_+(f) \geqq 1$ (see Cassels [1], Ch. II). Dumir [4] proved that we have no λ_+ in the open interval (96/25, 4). This is a special case of the following theorem:

Theorem 1. *There is no λ_+ in the open interval*

$$\left(\left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2}, \left(\frac{k+1}{k}\right)^2\right), \qquad k = 1, 2, 3, \ldots.$$

The next theorem implies that in a certain sense Theorem 1 cannot be improved.

Theorem 2. *Let*

$$u_k(x, y) = x^2 + \frac{k^2+k+2}{k^2+2k+2} xy - \frac{k^2+k+2}{k(k^2+2k+2)} y^2$$

and

$$v_k(x, y) = x^2 + \frac{k-1}{k} xy - \frac{1}{k} y^2.$$