Weighted approximation, Mergelyan's theorem and quasi-analytic classes

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1. Introduction

It is well known that the classical Hadamard problem on the characterization of quasi-analytic classes of functions admits several equivalent formulations and solutions in terms of holomorphic functions, asymptotic series, divergent series, etc. (See [4] and [7].) Some of the techniques employed to attack this problem proceeded from the weighted approximation theory which originated with S. Bernstein. More precisely, from the very beginning of weighted approximation theory, there was an interdependence with quasi-analytic function theory. For applications of weighted approximation theory to the Hadamard problem, see, for instance, [2], [3], [5] and [6]. On the other hand, the Denjoy—Carleman theorem on quasi-analyticity has been widely used to get results in weighted approximation theory. In fact, almost all these results contain a hypothesis implying that some weight is fundamental in the sense of S. Bernstein by using the above theorem. See, for instance, [7], [9], [10], [11] and [12]. Furthermore, the classical problem which consisted of characterizing the fundamental weights on the real line was completely solved by S. Mergelyan [8]. The key to Mergelyan's solution is a result which is appropriate for approximation, not only for the real line, but also for closed nowhere dense sets in the complex plane.

Our purpose here is twofold. First, we give a simple proof of Mergelyan's theorem (Theorem 1). Then, by using this result we establish that quasi-analyticity is equivalent to the fact that some weight is fundamental (Theorem 2). So, in some sense, we cement the interdependence mentioned above. As an application, we get another solution of Hadamard's problem (Theorem 3). All this is done by using

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