A p-extremal length and p-capacity equality

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1. Introduction

Let G be a domain in the compactified euclidean n-space $\bar{R}^n = R^n \cup \{\infty\}$, let E and F be disjoint non-empty compact sets in the closure of G. We associate two numbers with this geometric configuration as follows. Let $M_p(E, F, G)$ be the p-modulus (reciprocal of the p-extremal length) of the family of curves connecting E and F in G. Let $\operatorname{cap}_p(E, F, G)$ be the p-capacity of E and F relative to G, defined as the infimum of the numbers $\int_G |\nabla u(x)|^p \, dm(x)$ where u is an ACL function in G with boundary values 0 and 1 on E and F, respectively. We show in this paper that $\operatorname{cap}_p(E, F, G) = M_p(E, F, G)$ whenever E and F do not intersect ∂G. This generalizes Ziemer's [7] result where he makes the assumption that either E or F contains the complement of an open n-ball.

We also obtain a continuity theorem (Theorem 5.9) for the p-modulus and a theorem (Theorem 4.15) on the kinds of densities that can be used in computing the p-modulus.

2. Notation

For $n \geqq 2$ we denote by $\bar{R}^n$ the one point compactification of R^n, euclidean n-space: $\bar{R}^n = R^n \cup \{\infty\}$. All topological considerations in this paper refer to the metric space $(\bar{R}^n, q)$ where q is the chordal metric on $\bar{R}^n$ defined by stereographic projection. If $A \subset \bar{R}^n$ then $\bar{A}$ and ∂A denote the closure and boundary of A, respectively. If $b \in \bar{R}^n$ and $B \subset \bar{R}^n$ then $q(b, B)$ denotes the chordal distance of b from B.

If $x \in R^n$ we let $|x|$ denote the usual euclidean norm of x. $B^n(x, r)$ denotes the open n-ball with center x and radius r. We write $B^n(1) = B^n(0, 1)$. If $x \in R^n$ and $A \subset R^n$ we let $d(x, A)$ denote the euclidean distance of x from A.

Lebesgue n-measure on R^n is denoted by m_n or by m if there is no chance for confusion. We let $\Omega_n = m_n(B^n(1))$.