Uniform convergence of random Fourier series

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1. Introduction

We study the uniform convergence of the random trigonometric series

$$\sum_{n=0}^{\infty} a_n \eta_n \cos(nt + \Phi_n) \tag{1.1}$$

where $\{\eta_n e^{i\Phi_n}\}$ is a sequence of independent complex valued random variables (η_n and Φ_n are real). Various additional conditions are put on $\{\eta_n e^{i\Phi_n}\}$ (and $\{a_n\}$) in the different results obtained. This work was motivated by our desire to prove the following theorem.

Theorem 1.1. *Let $\{\eta_n e^{i\Phi_n}\}$ be independent symmetric complex valued random variables, $E|\eta_n|^2=1$ and $\liminf_{n\to\infty} E|\eta_n|>0$. Let $\{a_n\}\in l^2$ and assume that a_n is non-increasing ($a_n\downarrow$). Then*

$$\sum_{n=2}^{\infty} \frac{\left(\sum_{k=n}^{\infty} a_k^2\right)^{1/2}}{n(\log n)^{1/2}} < \infty \tag{1.2}$$

is a necessary and sufficient condition for the uniform convergence a.s. of the series (1.1).

The sufficient part of this theorem was obtained by Salem and Zygmund [7] in the case where Φ_n is a real number and $\{\eta_n\}$ a Rademacher sequence (a Rademacher sequence is a sequence of independent random variables $\{\varepsilon_n\}$ where $\varepsilon_n=\pm 1$ each with probability 1/2) and extended to independent symmetric $\{\eta_n e^{i\Phi_n}\}$ by Kahane [4]. In fact for sufficiency neither the condition $a_n\downarrow$ nor $\liminf_{n\to\infty} E|\eta_n|>0$ is needed, on the other hand symmetry is not needed for necessity. Theorem 1.1 was obtained for random trigonometric series that are also stationary Gaussian

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