Weak sequential convergence in the dual of a Banach space does not imply norm convergence

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We shall prove that for every infinite-dimensional Banach space E there is a sequence in E', the dual space, which tends to 0 in the weak topology $\sigma(E', E)$ but not in the norm topology. This is well known for separable or reflexive Banach spaces. See also [3] for other examples. The theorem has its main applications in the theory of holomorphic functions on infinite-dimensional topological vector spaces (TVS).

Let l^∞ be the Banach space of all complex-valued, bounded functions on the natural numbers $\mathbf{N}$; $z=(z_j)_{j=1}^\infty$ denotes a point in l^∞. Let c_0 be the Banach space $c_0=\{z\in l^\infty;\ z_j\to 0 \text{ as } j\to\ldots\infty\}$, $c=\{z\in l^\infty;\ \lim_{j\to\infty} z_j \text{ exists}\}$ and $l^1=\{z\in c_0,\ \sum_{j=1}^\infty |z_j|<<\infty\}$. Let $L(F, F_1)$ denote the set of all bounded linear mappings from F into F_1 and let $\mathfrak{H}(F)$ denote the set of Gâteaux-analytic, locally bounded functions on F, where F and F_1 are locally convex TVS. See [5]. A set $B\subset F$ is called *bouding* if $\sup_{z\in B} |f(z)|<\infty$ for every $f\in\mathfrak{H}(F)$. Put $\mathfrak{H}_b(F)=\{f\in\mathfrak{H}(F); f \text{ is bounded on bounded subsets of } F\}$.

Theorem. *To every infinite-dimensional Banach space E there exist $\varphi_j\in E'$ such that $\|\varphi_j\|=1$ and $\lim_{j\to\infty}\varphi_j(z)=0$ for every $z\in E$.*

Corollary 1. *No neighbourhood of $0\in F$, where F is a locally convex* TVS, *is a bounding set.*

Proof. See [2].

Corollary 2. $\mathfrak{H}_b(E)\neq\mathfrak{H}(E)$ *for very infinite-dimensional Banach space E.*

Proof. Se [2].

Proof of the Theorem. Let $F\subset E$ be a separable, infinite-dimensional subspace. From [1] and [2] it follows that there are $z^{(j)}\in F$ and $\psi_j\in E'$ such that $\|\psi_j\|=1$, $\|z^{(j)}\|=1$, $\psi_j(z^{(j)})=1$ and $\lim_{j\to\infty}\psi_j(z)=0$ for every $z\in F$. Let $\psi\in L(E, l^\infty)$ be the