Invariant Lorentzian orders on simply connected Lie groups

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0. Introduction

The study of the causal structure of space-time in the theory of relativity led a series of authors to the study of partially ordered Lie groups (cf. [Pa 81, 84], [Ol 81, 82], [Gu 76], [Le 84], [Se 76], [Vi 80]). We restrict ourselves to orders that satisfy a certain compatibility condition between the order and the algebraic structure:

Definition 0.1. Let G be a group and $\leqq$ be a partial order on G, then $\leqq$ is called an *invariant order* if the following monotonicity law holds:

(M) $\quad g_1 \leqq g_2$ then $hg_1k \leqq hg_2k$ for all $h, k, g_1, g_2 \in G$.

We can associate with an order $\leqq$ on G its *positive cone* $P_{\leqq}=\{g\in G\colon g\geqq \mathbf{1}\}$ where $\mathbf{1}$ is the identity of G. It is an easy exercise to see that an invariant order is completely determined by its positive cone:

Remark 0.2. Let G be a group and P a subset of G. Define a relation $\mathcal{R}\subset G\times G$ on G by setting $g_1\mathcal{R}g_2$ if and only if $g_2g_1^{-1}\in P$. Then $\mathcal{R}$ is an invariant order with $P_{\mathcal{R}}=P$ if and only if the following conditions hold:

(i) $P\cap P^{-1}=\{e\}$.
(ii) $PP\subset P$, i.e. P is a semigroup.
(iii) $gPg^{-1}\subset P$ for all $g\in G$. $\square$

An invariant order $\leqq$ on a group G will be called *directed* if for any two g_1, $g_2\in G$ there exist $h_1, h_2\in G$ such that $h_1\leqq g_1$, $g_2\leqq h_2$. Again we can translate this property of the order $\leqq$ into a property of its positive cone $P_{\leqq}$ by a standard argument:

Remark 0.3. Let G be a group and $\leqq$ be an invariant ordering of G. Then the