Isometric embedding of a smooth compact manifold with a metric of low regularity

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1. Introduction

Let X be a compact C^∞ manifold of dimension $n>1$ with a C^k Riemannian metric G. By an isometric embedding of X in $\mathbf{R}^N$ we mean an injective function $U\in C^1(X, \mathbf{R}^N)$ which induces the given metric, that is

$$(dU, dU) = G. \tag{1.1}$$

Nash [7] proved that if $G\in C^0$ there is an isometric embedding $U\in C^1(X, \mathbf{R}^N)$ provided that $N\geqq n+2$ and that there is a differentiable embedding of X in $\mathbf{R}^N$, in particular if $N\geqq 2n$. Nash also indicated that the condition $N\geqq n+2$ could be weakened to $N\geqq n+1$, which was proved by Kuiper [6]. It should be observed that (1.1) in local coordinates means $n(n+1)/2$ equations for N variables. For $G\in C^k$, $k\geqq 3$, Nash [8] also showed that there is an embedding $U\in C^k(X, \mathbf{R}^N)$ if $N\geqq n(3n+11)/2$. The condition on N has been improved for smooth metrics to $N\geqq n(n+1)/2+3n+5$ by Gromov and Rokhlin [3], who also gave lower estimates for the embedding dimension of the same order of magnitude for $k\geqq 2$. This result of Nash was extended by Jacobowitz [5] to Hölder classes H^a with $a>2$, and he also showed that there are metrics $G\in H^\beta$, $\beta>2$, such that (1.1) has no solution $U\in H^\alpha(X, \mathbf{R}^N)$, $\alpha>\beta$, for any N.

The result of Nash—Kuiper shows in particular that there is always a local embedding of X in $\mathbf{R}^{n+1}$. Borisov [1] has announced that if G is analytic there is a local isometric embedding $U\in H^\alpha(X, \mathbf{R}^{n+1})$ with any $\alpha<1+1/(n^2+n+1)$. Thus α is close to 1 if n is large. The low regularity seems to be caused by the demand for a low codimension, for by permitting large values of N we shall prove

Theorem 1.1. *If $G\in H^\beta$, $0<\beta\leqq 2$, then the equation* (1.1) *has a solution $U\in H^\alpha(X, \mathbf{R}^N)$ if $\alpha<1+\beta/2$ and N is sufficiently large. On the other hand, if $0\leqq\beta<2$ the set of all $G\in H^\beta$ for which* (1.1) *has a solution $U\in H^\alpha(X, \mathbf{R}^N)$ with $\alpha>1+\beta/2$ is of the first category.*