Excursions in Brownian motion

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Dedicated to the Memory of Paul Lévy

1. Introduction

Paul Lévy initiated his profound study of Brownian motion on the line in his article [10] of 1939 and expounded it in one chapter of his book [11]. The article contained a wealth of ideas that inspired a generation of research. A pivot in his approach is the time set when the Brownian path takes the value zero. His idea was to use this set to partition the time axis, so as to resolve the behavior of the path into two parts: the location of the zeros, and the motion in a zero-free interval. This idea is a natural extension of the consideration of successive entrances into a fixed state in a discrete time recurrent Markov chain. But since the zeros of a Brownian path do not form a well-ordered set in the natural order of the line, the execution of the intuitive ideas is not easy. Indeed, Lévy had recourse to another time set, that when the path is surpassing its previous maximum, which he found to be of the same stochastic structure as the zero-set. He based his analysis on the new set, which also led him to the discovery of local time. Despite this brilliant detour, it turned out that a direct attack on the zeros brought quick success, as shown in Theorem 1 below. Moreover, once the crucial calculations have been made, the rest of the denouement follows the pattern of last-exit phenomenon now familiar in Markov processes. The analogy may be pushed further by treating "zero" as a unique boundary point. There is much to be gained from the analogy even from the analytic point of view. For many explicit expressions reveal themselves to be the results of juxtapositions and cancellations of basic probabilistic quantities, and their combinations and transformations are facilitated by the probabilistic insight. This is the gist of the contents of § 2, which may be regarded as a re-stumping of Lévy's old ground with a new guide.

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