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Entire curves avoiding given sets in $\mathbf{C}^n$

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Let $F\subset\mathbf{C}^n$ be a proper closed subset of $\mathbf{C}^n$ and $A\subset\mathbf{C}^n\setminus F$ be at most countable, $n\geq 2$. The aim of this note is to discuss conditions for F and A, under which there exists a holomorphic immersion (or a proper holomorphic embedding) $\varphi:\mathbf{C}\to\mathbf{C}^n$ with $A\subset\varphi(\mathbf{C})\subset\mathbf{C}^n\setminus F$. Our main tool for constructing such mappings is Arakelian's approximation theorem (cf. [3] and [10]).

The first result is a generalization of the main part of Theorem 1 in [7]. More precisely, we prove the following result.

Proposition 1. *Let F be a proper convex closed set in $\mathbf{C}^n$, $n\geq 2$. Then the following statements are equivalent:*

(i) *either F is a complex hyperplane or it does not contain any complex hyperplane;*

(ii) *for any integer $k\geq 1$ and any two sets $\{\alpha_1,\dots,\alpha_k\}\subset\mathbf{C}$ and $\{a_1,\dots,a_k\}\subset\mathbf{C}^n\setminus F$, there exists a proper holomorphic embedding $\varphi:\mathbf{C}\to\mathbf{C}^n$ such that $\varphi(\alpha_j)=a_j$, $1\leq j\leq k$, and $\varphi(\mathbf{C})\subset\mathbf{C}^n\setminus F$.*

(iii) *the same as* (ii) *but for $k=2$.*

The equivalence of (i) and (iii) follows from the proof of Theorem 1 in [7]. For the convenience of the reader we repeat here the main idea of the proof of (iii) $\Rightarrow$ (i). Observe that condition (iii) implies that the Lempert function of the domain $D:=\mathbf{C}^n\setminus F$ is identically zero, i.e.

$$\tilde{k}_D(z,w):=\inf\{\alpha\geq 0:\text{there is } f\in\mathcal{O}(\Delta,D) \text{ with } f(0)=z \text{ and } f(\alpha)=w\}=0,$$

$z,w\in D$, where Δ denotes the open unit disc in $\mathbf{C}$. In the case when condition (i) is not satisfied we may assume (after a biholomorphic mapping) that $F=A\times\mathbf{C}^{n-1}$, where the closed convex set A, properly contained in $\mathbf{C}$, contains at least two points. Applying standard properties of $\tilde{k}$, we have $\tilde{k}_D(z,w)=\tilde{k}_{\mathbf{C}\setminus A}(z',w')$, where $(z,w)=((z',z''),(w',w''))\in D$. Since $\tilde{k}_{\mathbf{C}\setminus A}$ is not identically zero we end up with a contradiction.