On fundamental solutions supported by a convex cone

ARNE ENQVIST

1. Introduction

Let $P(D)$ be a partial differential operator in $\mathbf{R}^n$ with constant coefficients and Γ a closed convex cone in $\mathbf{R}^n$. Thus we assume that $x, y \in \Gamma$ and $s, t \geq 0$ implies that $sx + ty \in \Gamma$. The problem discussed here is to decide when $P(D)$ has a fundamental solution with support in Γ.

When Γ is a proper cone, that is, when Γ contains no straight line, this condition means precisely that $P(D)$ is hyperbolic with respect to the supporting planes of Γ which meet Γ only at the origin (Gårding [4], see also Hörmander [5, Theorem 5.6.2]). In the other extreme case where Γ is a half space sufficient conditions were given long ago by Petrowsky (see Gelfand — Shilov [2]), and a complete answer to the question was obtained by Hörmander [6].

In general the intersection $\Gamma \cap (-\Gamma) = W$ is a linear subspace and $x \in \Gamma$ implies $x + y \in \Gamma$ for every $y \in W$. This shows that Γ is the inverse image in $\mathbf{R}^n$ of the image V of Γ in $\mathbf{R}^n/W$ under the quotient map. It is clear that V is a proper cone. We shall use the notations $n' = \dim W$, $n'' = n - n'$ and coordinates $x = (x', x'')$ such that W is defined by $x'' = 0$. Also for $n' > 0$ and $n'' > 1$ sufficient conditions for the existence of a fundamental solution of $P(D)$ with support in Γ, analogous to those of Petrowsky for $n'' = 1$, have been given by Gindikin [3]. We shall extend these in the direction suggested by the technique used by Hörmander [6]. However, when $n'' > 1$ there are polynomials such that $P(\zeta', D'')$ is not hyperbolic for any ζ'. This introduces a new difficulty and in consequence of this the result is far from complete.

In Section 2 we investigate the general necessary conditions. The methods used are very close to those of Hörmander [6]. In the hyperbolic case the principal part plays a very important role. (See L. Svensson [9].) Here the principal part does not give so much information about the polynomial, and we have not been able to find any substitute. However, in Section 3 we study some stability properties of the necessary conditions which allow us to carry them over to various polynomials related to the behavior of P at infinity.