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On Gröchenig, Heil, and Walnut's proof of the local three squares theorem

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It is plain that the positive (and nontrivial) part of Theorem 5.1 of the preceding paper by K. Gröchenig, C. Heil, and D. Walnut [1] is equivalent to the following theorem. (We keep the notation from [1] with the following slight exception: μ_k denotes the characteristic function of the interval $[-r_k, r_k]$, and ν_k is the characteristic function of the d-dimensional cube $[-r_k, r_k]^d$.)

Theorem. *Suppose $0<r_1<r_2$ and $r_1/r_2 \notin \mathbf{Q}$, and set $R=r_1+r_2$. Then the set of functions of the form $(g_1\mu_1)*\mu_2+\mu_1*(g_2\mu_2)$, with $g_1, g_2 \in L^2(-R,R)$, is dense in $L^2(-R,R)$.*

This observation underlies the proof of Theorem 5.1 of [1]. Below I will give a more direct proof of the theorem just stated. A d-dimensional extension, equivalent to the local three "squares" theorem in dimension $d>1$ (cf. Theorem 6.1 of [1]), will be obtained as a corollary of this theorem.

Proof. It is enough to consider linear combinations of $(g_1\mu_1)*\mu_2$, with $g_1(t)=e^{i\pi kt/r_1}$, and $\mu_1*(g_2\mu_2)$, with $g_2(t)=e^{i\pi kt/r_2}$, k denoting an arbitrary integer. Taking Fourier transforms, we see that the question is whether the linear span of the functions

$$\frac{G(t)}{t(t-\lambda_{j,k})},$$

with $\lambda_{j,k}=k/2r_j$ and $G(t)=\sin(2\pi r_1 t)\sin(2\pi r_2 t)$, is dense in the Paley–Wiener space PW_{2R}. This is answered in two steps.

First we prove that $G(t)/t$ belongs to the closed span of these functions: Choosing $a_k>0$ such that $\sum_k a_k=1$ and $\sum_k a_k^2=\varepsilon$, we obtain

$$\int_{\mathbf{R}}\left|\frac{G(t)}{t}-\sum_k \frac{a_k\lambda_{1,k}G(t)}{t(\lambda_{1,k}-t)}\right|^2 dt=\int_{\mathbf{R}}\left|\sum_k \frac{a_k G(t)}{\lambda_{1,k}-t}\right|^2 dt \le 2r_1\pi^2\varepsilon,$$