Local solvability of second order differential operators on nilpotent Lie groups

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1. Introduction

The main objective of this work is to establish sufficient conditions for the local solvability of certain left invariant differential operators on a nilpotent Lie group G. The operators to be considered are of the form

$$L = \sum_{j,k} a_{jk} X_j X_k + \sum_{p,q} C_{p,q} [X_p, X_q] \tag{1.1}$$

where $\{X_j\}$ is a set of generators for the Lie algebra $\mathfrak{G}$ of G, (a_{jk}) is a positive definite quadratic form, and each C_{pq} is a complex constant. If the C_{pq} are all real, Hörmander's criterion [16] implies that L is hypoelliptic and locally solvable. However, if the C_{pq} are imaginary both hypoellipticity and local solvability may fail as happens for instance when G is the Heisenberg group. Nevertheless, we will show that for many interesting classes of groups, all operators of the form (1.1) are locally solvable, even when not hypoelliptic.

This investigation has its origin in the author's attempt to understand the significance of the criterion for solvability of the Lewy equation, as well as the associated boundary Laplacian equation, given by Greiner, Kohn, and Stein [7]. (Similar results had previously been obtained in a different context by Sato, Kawai, and Kashiwara [30].) In [7], $\mathfrak{G}$ is the Heisenberg algebra, say of dimension three, and $L = X_1^2 + X_2^2 + i[X_1, X_2]$. Among other results it is proved that the equation $Lu = f$, f smooth, has a local smooth solution u at x_0 if and only if the orthogonal projection of f onto the L^2 kernel of L is real analytic near x_0. This result suggests a close relationship between the existence of a nontrivial global L^2 kernel for L^t and the local nonsolvability of L (see [2]).

Any unitary irreducible representation π of G acting on a Hilbert space $\mathscr{H}$ determines a corresponding representation, again denoted π of $\mathfrak{G}$ on $\mathscr{H}$; hence $\pi(L)$ is also defined as an operator on $\mathscr{H}$. For the Heisenberg group, the existence