Algebraic surfaces containing an ample divisor of arithmetic genus two

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Introduction

After some progress was made in the study of the adjunction process [19], these last few years many papers [12], [15], [16] have appeared on the classical subject of classifying projective algebraic surfaces whose general hyperplane section has a given genus.

A more general and intrinsic version of this problem can be stated as follows: classify all pairs $(S, \mathscr{L})$ where S is a smooth complex projective algebraic surface and $\mathscr{L}$ an ample line bundle on S whose arithmetic genus $g(\mathscr{L})=1+\frac{1}{2}(\mathscr{L}^2+\mathscr{L}K_S)$ is a given number g. Of course $g(\mathscr{L})\geqq 0$. In the cases $g=0$ and $g=1$ this classification is known [13] (see also [7], [8]) and, in a sense, gives nothing new with respect to the classical case where $\mathscr{L}$ is assumed a very ample line bundle.

In the case $g=2$ the situation is quite different since some meaningful new pairs appear with respect to the classical case, e.g. the pair $(J(C), \mathcal{O}_{J(C)}(C))$ defined by a smooth curve C of genus two embedded in its Jacobian $J(C)$ and the pair $(\Sigma, \pi^*\mathcal{O}_{\mathbf{P}^2}(1))$ where $\pi\colon \Sigma\to\mathbf{P}^2$ is a double cover branched along a smooth sextic.

In this paper we give a classification of the polarized pairs $(S, \mathscr{L})$ with $g(\mathscr{L})=2$. Just a few words about what we mean by classifying pairs $(S, \mathscr{L})$. First of all S is classified birationally, according to the Enriques—Kodaira classification. As far as the line bundle $\mathscr{L}$ is concerned, since the definition of $g(\mathscr{L})$ involves numerical characters only, it seems reasonable to classify $\mathscr{L}$ up to numerical equivalence. The results we find are too complicated to be outlined here, so after noticing that they are summarized in various tables section by section, we use this introduction to point out the defects of our classification. Indeed here we find necessary conditions for polarized pairs $(S, \mathscr{L})$ to exist, but we are not always able to decide whether all the pairs with the characters we find do exist. However pairs $(S, \mathscr{L})$ where S is a minimal model of Kodaira dimension $\varkappa(S)\neq 1$ really occur.