# Determination of invariant convex cones in simple Lie algebras

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## 1. Introduction

In this paper, the determination of all convex cones in a real simple Lie algebra invariant under the adjoint group will be essentially reduced to an abelian problem. Specifically, if $\mathbf{h}$ is a compact Cartan subalgebra in the simple Lie algebra $\mathbf{g}$, the mapping $C \to C \cap \mathbf{h}$ is shown one-to-one from the class of (open or closed) invariant convex cones $C$ in $\mathbf{g}$, onto an explicitly described class of cones in $\mathbf{h}$ invariant under the Weyl group. All such cones $C$ in $\mathbf{g}$ have open dense interiors, and each such interior element, whether regular or not, is contained in a unique maximal compact subalgebra. The orthogonal projection of the orbit of such an interior element onto a compact Cartan subalgebra is determined explicitly, extending to these noncompact orbits known results for projections of compact group orbits. The above correspondence $C \to C \cap \mathbf{h}$ is shown to preserve the duality relation between cones, and the class of self-dual cones in the classical algebras corresponding to convex quadratic cones in the compact Cartan subalgebras is determined.

It is well known that the Poincaré group, the symmetry group of Minkowski space, contains a four-dimensional invariant semigroup, that of all vector displacements into the "future". This semigroup is the exponential of a corresponding invariant convex cone in the Lie algebra, which is precisely the cone of generators that are carried into nonnegative self-adjoint operators by infinitesimal unitary representations of "positive energy", such as those associated with certain hyperbolic partial differential equations (for example, Maxwell's equations).

This situation is not peculiar to the Poincaré group, but is applicable to a variety of other groups. For example, the universal cover $\tilde{G}$ (locally identical

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