Lower bounds for pseudo-differential operators

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0. Introduction

Let A be a classical scalar pseudo-differential operator of order m (cf. Kohn — Nirenberg [7]) in an open subset Ω of R^n. We are interested in estimates of A from below of the form

$$\operatorname{Re}(Au, u) \geq C|u|^2_{(s)}, \quad u \in C_0^\infty(K) \tag{0.1}$$

where K is a compact subset of Ω and $|u|_{(s)}$ is the norm of u in the space $H_{(s)}$ of functions with derivatives of order s in L^2 and $(v, u) = \int v\bar{u}\,dx$. If $s \geq m/2$ the estimate is always true for some C since (Au, u) is continuous in $H_{(m/2)}$. On the other hand, if $s < m/2$ it is easy to see that (0.1) implies

$$\operatorname{Re} a_m(x, \xi) \geq 0 \tag{0.2}$$

where a_m is the principal symbol of A. In the opposite direction Gårding [3] proved that if (0.2) is valid, then we can for every $\varepsilon > 0$ and every s find a constant $C = C(K, \varepsilon, s)$ such that

$$\operatorname{Re}(Au, u) + \varepsilon|u|^2_{(\mu)} \geq C|u|^2_{(s)}, \quad u \in C_0^\infty(K) \tag{0.3}$$

if $\mu = m/2$. A simple modification of the proof gives the same result for any $\mu > (m-1)/2$. In fact if A satisfies (0.2) and $m/2 \geq \mu > (m-1)/2$ then we can write

$$(A + A^*)/2 + \varepsilon(1 + |D|^2)^\mu = P^*P + Q$$

where P and Q are pseudo-differential operators in Ω and the order of Q does not exceed $m - 1$.

However the situation becomes more complex when $\mu = (m-1)/2$. It was proved by Hörmander [5] that (0.2) does imply that (0.3) is valid for some $\varepsilon > 0$, but to have (0.3) for every $\varepsilon > 0$ we must clearly in addition to (0.2) place a restriction on the terms in A of order $m - 1$. In this paper we shall study necessary and sufficient conditions on A for (0.3) to be valid for every $\varepsilon > 0$ when