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Homogenization and boundary layers

by

David Gérard-Varet
Université Denis Diderot Paris 7
Paris, France

Nader Masmoudi
New York University
New York, NY, U.S.A.

This paper deals with the homogenization of elliptic systems with a Dirichlet boundary condition, when the coefficients of both the system and the boundary data are ε-periodic. We show that, as $\varepsilon \to 0$, the solutions converge in L^2 with a power rate in ε, and identify the homogenized limit system. Due to a boundary layer phenomenon, this homogenized system depends in a non-trivial way on the boundary. Our analysis answers a longstanding open problem, raised for instance in [6]. It substantially extends previous results obtained for polygonal domains with sides of rational slopes as well as our previous paper [14], where the case of irrational slopes was considered.

1. Introduction

This paper is about the homogenization of elliptic systems in divergence form

$$-\nabla \cdot \left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u \right)(x) = 0, \quad x \in \Omega, \tag{1.1}$$

set in a bounded domain Ω of $\mathbb{R}^d$, $d \geqslant 2$, with oscillating Dirichlet data

$$u(x) = \varphi\left(x, \frac{x}{\varepsilon}\right), \quad x \in \partial\Omega. \tag{1.2}$$

As is customary, $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is a family of functions of $y \in \mathbb{R}^d$, indexed by $1 \leqslant \alpha, \beta \leqslant d$, with values in the set of $N \times N$ matrices. Also, $u = u(x)$ and $\varphi = \varphi(x, y)$ take their values in $\mathbb{R}^N$. We recall, using Einstein's convention for summation, that for each $1 \leqslant i \leqslant N$,

$$\left(\nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right) \nabla u \right)_i (x) := \partial_{x_\alpha} \left[A_{ij}^{\alpha\beta} \left(\frac{\cdot}{\varepsilon}\right) \partial_{x_\beta} u_j \right](x).$$

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