The Diophantine Equation $x^2 + 7 = 2^n$

By T. Nagell

In Vol. 30 of the Norsk Matematisk Tidsskrift, pp. 62–64, Oslo 1948, I published a proof of the following theorem:\(^1\)

When $x$ is a positive integer, the number $x^2 + 7$ is a power of 2 only in the following five cases: $x = 1, 3, 5, 11, 181$.

Since prof. L. J. Mordell drew my attention to a paper by Chowla, Lewis and Skolem in the Proceedings of the American Mathematical Society, Vol. 10 (1959), p. 663–669, on the same subject, I consider it necessary to publish in English my proof of 1948 which is quite elementary.

The problem consists in determining all the positive integers $x$ and $y$ which satisfy the relation

$$\frac{1}{4} (x^2 + 7) = 2^y. \quad (1)$$

It is evident that the difference of two integral squares $u^2$ and $v^2$ is equal to 7 only for $u^2 = 16$ and $v^2 = 9$. Hence we conclude that the exponent $y$ in (1) can be even only for $y = 2$ and $x = 3$. Thus we may suppose that $y$ is odd and $\geq 3$.

Passing to the quadratic field $K(\sqrt[4]{-7})$, in which factorization is unique, we get from (1)

$$\frac{x + \sqrt{7}}{2} = \left(\frac{1 + \sqrt{-7}}{2}\right)^y, \quad (2)$$

whence

$$\left(\frac{1 + \sqrt{-7}}{2}\right)^y - \left(\frac{1 - \sqrt{-7}}{2}\right)^y = \pm \sqrt{-7}. \quad (3)$$

Considering this equation modulo

$$\left(\frac{1 - \sqrt{-7}}{2}\right)^2 = \frac{-3 + \sqrt{-7}}{2},$$

we get, since $y$ is odd and $\geq 3$, and since

$$\left(\frac{1 + \sqrt{-7}}{2}\right)^2 = \frac{-3 + \sqrt{-7}}{2} \equiv 1 \pmod{\frac{-3 + \sqrt{-7}}{2}},$$

\(^1\) The theorem is set as a problem in my Introduction to Number Theory, Stockholm and New York 1951 (Problem 165, p. 272).