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The Diophantine Equation $x^2+7=2^n$

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In Vol. 30 of the Norsk Matematisk Tidsskrift, pp. 62–64, Oslo 1948, I published a proof of the following theorem:[1]

When x is a positive integer, the number x^2+7 is a power of 2 only in the following five cases: $x=1, 3, 5, 11, 181$.

Since prof. L. J. Mordell drew my attention to a paper by Chowla, Lewis and Skolem in the Proceedings of the American Mathematical Society, Vol. 10 (1959), p. 663–669, on the same subject, I consider it necessary to publish in English my proof of 1948 which is quite elementary.

The problem consists in determining all the positive integers x and y which satisfy the relation

$$\frac{1}{4}(x^2+7)=2^y. \tag{1}$$

It is evident that the difference of two integral squares u^2 and v^2 is equal to 7 only for $u^2=16$ and $v^2=9$. Hence we conclude that the exponent y in (1) can be even only for $y=2$ and $x=3$. Thus we may suppose that y is odd and $\geqq 3$.

Passing to the quadratic field $K(\sqrt{-7})$, in which factorization is unique, we get from (1)

$$\frac{x\pm\sqrt{-7}}{2}=\left(\frac{1+\sqrt{-7}}{2}\right)^y, \tag{2}$$

whence

$$\left(\frac{1+\sqrt{-7}}{2}\right)^y-\left(\frac{1-\sqrt{-7}}{2}\right)^y=\pm\sqrt{-7}. \tag{3}$$

Considering this equation modulo

$$\left(\frac{1-\sqrt{-7}}{2}\right)^2=\frac{-3-\sqrt{-7}}{2},$$

we get, since y is odd and $\geqq 3$, and since

$$\left(\frac{1+\sqrt{-7}}{2}\right)^2=\frac{-3+\sqrt{-7}}{2}\equiv 1\left(\operatorname{mod}\frac{-3-\sqrt{-7}}{2}\right),$$

[1] The theorem is set as a problem in my *Introduction to Number Theory*, Stockholm and New York 1951 (Problem 165, p. 272).