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Multi-dimensional integral limit theorems

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1. Introduction

Let $X=(X_1, \ldots, X_k)$ be a random vector (r.v.) in the k-dimensional Euclidean space R_k, $k>1$, with zero mean and non-singular covariance matrix M. Further, let $X^{(1)}, \ldots, X^{(n)}$ be a sequence of independent r.v.'s in R_k with the same distributions as X. Then the normed sum $Y_n=n^{-\frac{1}{2}}\sum_{\nu=1}^{n}$ is approximately normally distributed with zero mean and covariance matrix M. Bergström [3] has shown that if $F_n(x)$, $x\in R_k$, is the d.f. of Y_n, and $\Phi(x)$ is the corresponding normal d.f. then, if the moments of the third order are finite:

$$|F_n(x)-\Phi(x)|\leqslant C\,n^{-\frac{1}{2}} \tag{1}$$

where C is a constant only depending on the moments of X. Esseen [8] has studied $F_n(A)=\int_A dF_n(x)$, where A is a closed sphere in R_k with its center in the origin: $A=\{x:|x|\leqslant a\}$ $(|x|=(x_1^2+\ldots+x_k^2)^{\frac{1}{2}})$ in the case $M=E_k$ (identity matrix of order $k\times k$). His result is that, if the moments of the fourth order are finite, then,

$$|F_n(A)-\Phi(A)|\leqslant C\,n^{-k/(k+1)}$$

Under the same condition, R. R. Rao [7] has announced without proof the result

$$|F_n(B)-\Phi(B)|\leqslant C\,n^{-1/2}(\log n)^{\beta}$$

where $\beta=(k-1)/2(k+1)$, valid uniformly for all convex Borel sets $B\subset R_k$, and also the expansion of $F_n(B)$ in powers of $n^{-1/2}$ given in Theorem 4, but with the remainder term $0(n^{-(s-2)/2}(\log n)^{(k-1)/2})$.

If the d.f. of X either has an absolutely continuous component or is of lattice type, it is possible to prove local limit theorems, that is, limit theorems for the density function of the absolutely continuous component of $F_n(x)$ or for probabilities corresponding to the lattice points of $F_n(x)$. By integrating (summing) the remainder terms in theorems of this type, A. Bikjalis [4, 5] has obtained integral limit theorems for arbitrary Borel sets and for arbitrary subsets of the lattice set of $F_n(x)$ respectively.

In the present paper, I shall prove two generalizations of (1) (Theorems 1 and 2) by a method which is entirely different from the one used by Bergström, who considers an expansion of $(F_n(x)-\Phi(x))*\Phi(x/\varepsilon)$ (convolution), together with an estimation of Weierstrass's singular integral. Theorems 3 and 4 give, as mentioned above, estimates of $F_n(B)$ for arbitrary Borel sets and for convex Borel sets respectively, when the moments of order r, $2<r\leqslant 3$, or of order s, $s\geqslant 3$, are finite.