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On the central limit theorem in R_k

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1. Introduction

Let $X^{(\nu)} = (X_1^{(\nu)}, \dots X_k^{(\nu)})$, $\nu = 1, 2, \dots n$, be a sequence of independent and identically distributed random vectors (r.v.'s) in R_k, $k > 1$, with zero mean and non-singular covariance matrix M. Then, according to the Central Limit Theorem, the normed sum $Y_n = n^{-\frac{1}{2}} \sum_{\nu=1}^{n} X^{(\nu)}$ is approximately normally distributed, with the same moments of the first and second orders as $X^{(1)}$. In the present paper, we shall consider the distribution of the norm $|Y_n| = (Y_{n1}^2 + \dots + Y_{nk}^2)^{\frac{1}{2}}$, and estimate the difference

$$P(|Y_n| \leqslant a) - \int_{|x| \leqslant a} d\,\Phi(x), \tag{1}$$

where $\Phi(x)$, $x = (x_1, \dots x_k)$ is the corresponding normal distribution function (d.f.) and $|x| = (x_1^2 + \dots + x_k^2)^{\frac{1}{2}}$. If the moments of the fourth order exist and if $M = E$ (unit matrix of order $k \times k$), then (Esseen [3])

$$|P(|Y_n| \leqslant a) - K_k(a^2)| \leqslant C n^{-k/(k+1)}, \tag{2}$$

where $K_k(x)$ is the d.f. of the χ^2-distribution with k degrees of freedom, and C is a finite constant, only depending on the moments of $X^{(1)}$. Here we shall study the difference (1) as a function of both n and a.

2. Convergence of characteristic functions

We introduce the d.f.'s $F(x)$ and $F_n(x)$ and the characteristic functions (ch.f.'s) $f(t)$ and $f_n(t)$ of $X^{(1)}$ and Y_n respectively. We have

$$f(t) = \int_{R_k} e^{i(t,x)}\, dF(x), \quad t = (t_1, \dots t_k), \quad (t, x) = \sum_{j=1}^{k} t_j x_j$$

and $f_n(t) = f^n(t/\sqrt{n})$. If the moment $\beta_r = E|X^{(1)}|^r < \infty$, r integer $\geqslant 3$, then $\log f(t)$ has the Taylor expansion

$$\log f(t) = -\tfrac{1}{2}(t, Mt) + \sum_{\nu=3}^{r} \frac{(\varkappa, it)^{\nu}}{\nu!} + o(|t|^r), \tag{3}$$