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Generalized hyperbolicity

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Introduction

Let $x=(x_1, x_2, \dots x_n)$ be coordinates in R^n with the scalar product $(x,x')=\sum_{j=1}^n x_j x_j'$ and the norm $|x|$. We define

$$D=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \dots \frac{1}{i}\frac{\partial}{\partial x_n}\right), \quad D^\alpha = \prod_{\alpha_k \neq 0}\left(\frac{1}{i}\frac{\partial}{\partial x_k}\right)^{\alpha_k} \quad \text{and} \quad |\alpha| = \sum_{k=1}^n \alpha_k,$$

where $\alpha=(\alpha_1, \alpha_2, \dots \alpha_n)$ is a multiindex with non-negative integer components. As in Schwartz [1], let $\mathcal{E}(O)$ be the Fréchet space of all infinitely differentiable functions on the open non-empty set $O \subset R^n$ topologized by the semi-norms $\sup_{x\in K}|D^\alpha \varphi(x)|$, where K is compact in O. A complex polynomial P is called hyperbolic with respect to $N \in R^n$ if $P(D)$ has a fundamental solution, locally in the dual space $\mathcal{E}'(R^n)$, with support in a cone $(x, N) \geqslant \varepsilon|x|$, $\varepsilon>0$. Let P_m be the principal part of P. Then, according to Gårding [1], P is hyperbolic with respect to N if and only if there is a constant C such that $P_m(N) \neq 0$ and $P(\xi + i\tau N) \neq 0$ when $\xi \in R^n$ and $\tau \leqslant -C$. We shall here investigate hyperbolicity in other suitable distribution spaces.

For fixed $d>1$ we consider in $\mathcal{E}(O)$ the quasi-norms

$$|\varphi, K|_{d,l} = \sup_{\substack{\alpha \\ x\in K}} l^{-|\alpha|}|\alpha|^{-|\alpha|d}|D^\alpha \varphi(x)|,$$

where $l>0$ and K is compact in O. Set

$$G(d, O) = \{\varphi;\ |\varphi, K|_{d,l} < \infty \text{ for every } l>0 \text{ and every compact } K \subset O\}$$

topologized by the semi-norms $|\varphi, K|_{d,l}$ (cf. Hörmander [1], p. 146). We observe some simple properties of $G(d, O)$ and related spaces. For instance, $G(d, O)$ is a Fréchet space and it contains non-vanishing functions with compact support exactly when $d>1$. Let H be the half space $(x, N) \geqslant 0$ and denote by $\overline{G_0(d, H)}$ the subspace of all functions in $G(d, R^n)$ supported by H. We prove that the mapping

$$P(D)\colon \overline{G_0(d, H)} \to \overline{G_0(d, H)}$$

is injective and has a continuous inverse if and only if there is a constant C such that $P_m(N) \neq 0$ and $P(\xi + i\tau N) \neq 0$ when $\xi \in R^n$ and $\tau \leqslant -C(1+|\xi|^{1/d})$. This is also the precise condition for the existence of a fundamental solution of $P(D)$, locally in the dual space $G'(d, R^n)$, with support in a cone $(x, N) \geqslant \varepsilon |x|$, $\varepsilon>0$. We call such polynomials d-hyperbolic with respect to N. When $d=\infty$, we get formally the hyperbolic