ARKIV FÖR MATEMATIK Band 4 nr 39

Communicated 25 October 1961 by A. Beurling and L. Carleson

Convergent Dirichlet series

By Henry Helson

1. Introduction

This note presents several results about Dirichlet series

$$f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (s=\sigma+it), \tag{1}$$

which are loosely connected by a common idea of proof, rather than by any formal dependence. Section 2 contains an elementary lemma, really a reformulation of the well-known formula expressing the abscissa of convergence of (1) in terms of the coefficients and exponents of the series. After this lemma, the sections can be read independently.

Section 3 treats the *convergence problem* for Dirichlet series: to determine the abscissa of convergence from properties of the function f. A famous and difficult theorem of Landau and Schnee gives *sufficient* conditions for (1) to converge in a region $\sigma>\sigma_0$. A restriction has to be put on the exponents λ_n as well as on f. Without any restriction on the exponents beyond the fundamental one

$$\lambda_1<\lambda_2<\ldots; \quad \lambda_n\to\infty, \tag{2}$$

we shall show (within the half-plane $\sigma>0$) that the series converges precisely as far to the left as $f(\sigma+it)/(\sigma+it)$ satisfies a uniform condition of growth, and *as a function of t is the Fourier transform of a function which is summable and bounded.* This theorem is superficial, and it does not lead easily to new convergence theorems. In particular, the new theorem does not obviously imply the theorem of Landau and Schnee. But the result shows *why* convergence theorems are difficult to prove: they amount to showing that functions with given properties are Fourier transforms, and simple criteria on which to decide do not exist.

In Section 4 we reconsider the *formula of Perron*:

$$\sum_{\lambda_n\leqslant x}{}' a_n=\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\sigma-i\omega}^{\sigma+i\omega}\frac{f(s)}{s}e^{ixs}\,ds. \tag{3}$$

The stroke on the sign of summation means, here and hereafter, that the final term is to be halved if the last λ_n is x. The formula holds for $\sigma>\sigma_1$, where (for the rest of the paper) σ_1 denotes the abscissa of convergence σ_c if that is positive, and 0 otherwise.