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# On some expansions of stable distribution functions

By HARALD BERGSTRÖM

## 1. Introduction

The function $e^{-t^\alpha}$ for any fixed value $\alpha$ in the interval $0 < \alpha < 1$ admits a unique representation.

$$e^{-t^\alpha} = \int_0^\infty e^{-xt} G'_\alpha(t)\, dx, \qquad 0 \leqq t \leqq \infty \tag{1}$$

where $G_\alpha(x)$ is a stable d. f. (distribution function) with $G_\alpha(0) = 0$.[1] P. HUMBERT[2] has formally given the expansion

$$G'_\alpha(x) = -\frac{1}{\pi} \sum_{k=1}^\infty \frac{(-1)^k}{k!} (\sin \pi \alpha k) \frac{\Gamma(\alpha k + 1)}{x^{\alpha k + 1}} \tag{2}$$

for $0 < \alpha < 1$, $x < 0$, which has later been rigorously proved by H. POLLARD.[3]
From (1) follows that $G_\alpha(x)$ has the characteristic function

$$\gamma_\alpha(t) = e^{-|t|^\alpha \left(\cos \frac{\pi \alpha}{2} - i \sin \frac{\pi \alpha}{2} \operatorname{sgn} t\right)}$$

(sgn: read signum). Now owing to P. LÉVY[4] the characteristic function of a stable d.f., when suitably normalized can be written in the form.

$$\gamma_{\alpha\beta}(t) = e^{-|t|^\alpha (\cos \beta - i \sin \beta \operatorname{sgn} t)}, \tag{3}$$

where

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[1] S. BOCHNER, Completely monotone functions of the Laplace operator for torus and sphere, Duke Math. J. vol. 3, 1937.

P. LÉVY, Théorie de l'addition des variables aléatoires, Gauthier-Willars (1937) pp. 94—97, 198—204.

Compare also H. POLLARD, The representation of $e^{-x^\lambda}$ as a Laplace integral, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 908—910.

[2] P. HUMBERT, Nouvelles correspondances symboliques, Bull. Soc. Math. France vol. 69 (1945) pp. 121—129.

[3] H. POLLARD loc. cit.

[4] P. LÉVY loc. cit.