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On a theorem of Hanner

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OLOF HANNER (See reference [4]) has shown that a separable metric space is an absolute neighborhood retract (ANR) for normal spaces if and only if it is both an ANR for separable metric spaces and an absolute G_δ. Using an example given in a recent paper of R. H. BING [1] we show (theorem 1) that if a metric space is an ANR for normal spaces it is separable, and that hence the hypothesis of separability in Hanner's theorem can be dropped. In the same paper Bing defined a class of spaces more restricted than normal called collectionwise normal. We show (theorem 2) that Hanner's theorem extends to non-separable metric spaces if normal is replaced by collectionwise normal. Moreover (corollary 1) this form of Hanner's theorem characterizes collectionwise normal spaces in the same way as Tietze's extension theorem characterizes normal spaces.

1. Given a class τ of spaces, a space Y belonging to τ is called an ANR_τ [respectively AR_τ] if $Y \in \tau$ and if every map f of a closed set A of a space X of class τ into the space Y can be extended to a map f_1 of an open set U, such that $A \subset U \subset X$, into Y [respectively, to a map f_1 of X into Y]. In particular ANR_n, ANR_{cn}, ANR_m and ANR_{sm} will mean absolute neighborhood retract for normal, collectionwise normal, metric and separable metric spaces respectively. If a class σ of spaces is contained in τ, if $Y \in \sigma$ and if Y is ANR_τ then clearly Y is also ANR_σ. (The above definition of ANR is equivalent in all cases considered below to the usual definition terms of retraction (See for example [4], theorem 3. 2) but we make no use of this equivalence.)

Theorem 1. *A metric space Y is ANR_n [respectively AR_n] if and only if it is ANR_m [respectively AR_m], separable and absolute G_δ.*

Proof. Sufficiency. If Y is separable and ANR_m it is ANR_{sm}. If it is also absolute G_δ then, by [4] theorem 4. 2, it is ANR_n.

Necessity. Let Y be metric and ARN_n. Suppose if possible that Y is not separable. Then there exists $\epsilon > 0$ and a non-countable subset B of Y such that each pair of points of B have distance $> \epsilon$. Bing ([1], page 184, example G) has shown that there exists a normal space X with a closed subset A of arbitrary non-countable cardinal number such that the subspace A has the discrete topology but no collection of mutually non-intersecting neighborhoods of the points of A exists. Choose for A the cardinal number of B and let f be a 1-1 map of A on B. Then $f: A \to Y$ is continuous and, since Y is ANR_n, can be extended to a map $f_1: U \to Y$ of a neighborhood U of A. The inverse images by f_1 of the $(\epsilon/2)$-neighborhoods of the points of B form a collection of non-intersecting neighborhoods in X of the points of A, which is impossible. Therefore Y is separable.