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A generalization of a theorem of Nagell

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1. As is well known, the coordinates of the curve

$$y^2 = x^3 - Ax - B \qquad (4A^3 - 27B^2 \neq 0) \tag{1}$$

can be represented by Weierstrass's $\wp$-function with the invariants $4A$ and $4B$:

$$\begin{cases} x = \wp(u; \quad 4A,\ 4B) \\ y = \frac{1}{2}\wp'(u; \quad 4A,\ 4B). \end{cases}$$

If u is commensurable with a period, the point (x, y) is called *exceptional.* In this case there is a natural number n, which makes nu a period, while $n'u$ is not a period, if $0 < n' < n$. This number n is called the *order* of the point (x, y). The point of order 1, corresponding to $u = 0$, is the infinite point of inflexion on the curve.

If A and B belong to a field Ω and if (x, y) is a point on (1), whose coordinates belong to Ω, we shall say that (x, y) is a *point in* Ω.

In 1935 T. NAGELL ([3], p. 8–15) proved the following theorem:

Theorem 1. — *If A and B are integers in $k(1)$ and if (x, y) is a finite exceptional point in $k(1)$ on the curve* (1), *then x and y are integers. If $y \neq 0$, then y^2 divides $4A^3 - 27B^2$.*

According to G. BILLING ([1], p. 120) this theorem remains true, if $k(1)$ is replaced by a quadratic or cubic field, but BILLING's proof is incomplete, since his lemmas do not say anything, if the order of the point (x, y) is a prime. BILLING's theorem is, however, contained in a generalization of theorem 1, which will be given in this paper.

2. We begin with a lemma on the function

$$x = \wp(u; \quad 4A,\ 4B).$$

It is known that if n is a natural number >1 and if nu is a period but u is not, then

$$\Psi_n(u) = 0,$$

where

$$\Psi_n(u) = \frac{\sigma(nu)}{[\sigma(u)]^{n^2}}.$$