ARKIV FÖR MATEMATIK Band 2 nr 5

Communicated 5 December 1951 by ARNE BEURLING

On the ideal structure of group algebras

By HENRY HELSON

Let G be a locally compact abelian group with dual group $\hat{G}$, and denote by $\mathfrak{L}$ the group algebra of G, consisting of the functions summable on G for Haar measure dx. The maximal regular ideals of $\mathfrak{L}$ are in correspondence with the points of $\hat{G}$; the functions belonging to such an ideal are exactly those whose Fourier transforms vanish at the given point. Now let $\mathfrak{J}$ be any closed ideal in $\mathfrak{L}$. The Tauberian theorem states that the Fourier transforms of the functions of $\mathfrak{J}$ have at least one common zero. Differently stated, $\mathfrak{J}$ is contained in at least one regular maximal ideal. The central problem about the ideal structure of $\mathfrak{J}$ is to determine under what conditions $\mathfrak{J}$ is the intersection of the regular maximal ideals which contain it; and this is the same as to decide whether $\mathfrak{J}$ necessarily contains all functions whose transforms vanish on the set of zeros of the transforms of $\mathfrak{J}$.

L. Schwartz [4] has given an example in the three-dimensional Euclidean group of a closed ideal which is actually smaller than the intersection of the maximal regular ideals containing it. Positive results can be given by restricting the ideals considered. We refer to [2] for these results and to references to the literature (the problem is there discussed in $\mathfrak{L}^\infty$ from the dual point of view). But our present point of departure is the fact that there are groups where distinct ideals determine the same set of zeros. We shall show in such a case that infinitely many ideals determine that set of zeros, and we can exhibit a few of them. The crucial tool is a theorem of Godement on unitary representations of abelian groups.

The reader should be familiar with the theory of Fourier transforms on abelian groups, in particular with the content of [1]. The theorem of Godement referred to is also proved in this paper.

For a closed set $\hat{E}$ in $\hat{G}$, let $\mathfrak{J}(\hat{E})$ be the closed ideal of summable functions on G whose transforms vanish on $\hat{E}$. Let $\mathfrak{J}_0(\hat{E})$ be the closure of the ideal of summable functions whose transforms vanish on some open set containing $\hat{E}$ (depending on the function). Then any closed ideal whose set of zeros is $\hat{E}$ lies between $\mathfrak{J}_0$ and $\mathfrak{J}$. We assume that $\mathfrak{J}_0$ and $\mathfrak{J}$ are distinct, and we are going to show that there are indeed a great many ideals between.

If $\mathfrak{U}$ and $\mathfrak{V}$ are closed ideals such that

$$\mathfrak{J}_0 \subset \mathfrak{U} \subset \mathfrak{V} \subset \mathfrak{J}$$