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On the solvability of the Diophantine equation $ax^2 + by^2 + cz^2 = 0$ in imaginary Euclidean quadratic fields

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§ 1. Introduction

LEGENDRE [1][1] has given the first proof of the following theorem:

Let a, b and c be three rational integers such that abc is square-free. Then the equation

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$

is solvable in rational integers x, y, z not all zero if and only if $-bc$, $-ca$, and $-ab$ are quadratic residues of a, b and c respectively, and a, b and c are not all of the same sign.

In DIRICHLET, Zahlentheorie [2], and in most other text-books of number theory, this theorem is proved by means of a method which we call the *index method*. See also T. NAGELL, Introduction to Number Theory [3].

Then the following question arises: In which algebraic fields is it possible to determine the necessary and sufficient conditions for the solvability of (1) by the index method?

An algebraic field is called *simple* when the number of ideal classes is $=1$. A simple field is said to be Euclidean when there is a Euclidean algorithm between every two integers α and β in the field, $\beta \neq 0$. See HARDY-WRIGHT: The Theory of Numbers [4].

The index method can only be applied to Euclidean fields, because it is based on an algorithm. In this paper we shall only consider the case of an imaginary Euclidean quadratic field. There are five such fields, namely $K(\sqrt{-1})$, $K(\sqrt{-2})$, $K(\sqrt{-3})$, $K(\sqrt{-7})$ and $K(\sqrt{-11})$.

In an imaginary field the condition "a, b and c not all of the same sign" has no meaning. We shall examine whether the other conditions are sufficient or not in these fields. TH. SKOLEM [5] has shown that they are in $K(\sqrt{-1})$ and $K(\sqrt{-3})$ but his method is rather different and we shall treat these fields too by means of the index method.

[1] Figures in [ ] refer to the Bibliography at the end of this paper.