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Behavior of solutions of linear second order differential equations

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1. Introduction. The present note is concerned with the differential equation

$$w'' = \lambda F(x) w, \tag{1.1}$$

where $F(x)$ is defined, positive and continuous for $0 \leqq x < \infty$, while λ is a complex parameter which, except in section 2, is not allowed to take on real values $\leqq 0$. We are mainly interested in qualitative properties of the solutions for large positive values of x including integrability properties on the interval $(0, \infty)$. In section 6 we shall discuss certain extremal problems for this class of differentiel equations.

The results are of some importance for the theory of the partial differential equations of the Fokker-Planck-Kolmogoroff type corresponding to temporally homogeneous stochastic processes. These applications will be published elsewhere. The results also admit of a dynamical formulation and interpretation. This will be used frequently in the following for purposes of exposition. With $x = t$, the equation

$$w'' = \lambda F(t) w \tag{1.2}$$

is the equation of motion in complex vector form of a particle

$$w = u + iv = r e^{i\theta} \tag{1.3}$$

under the influence of a force of magnitude

$$|P| = \varrho F(t) r, \quad \lambda = \varrho e^{i\varphi} = \mu + i\nu, \tag{1.4}$$

making the constant angle φ with the radius vector. We can also write the equations of motion in the form

$$r'' - r(\theta')^2 = \mu F(t) r, \tag{1.5}$$

$$\frac{d}{dt}[r^2 \theta'] = \nu F(t) r^2, \tag{1.6}$$

where the left sides are the radial and the transverse accelerations respectively.