Communicated 6 June 1951 by F. Carlson and T. Nagell

On the Diophantine equation $u^2 - Dv^2 = \pm 4N$

By Bengt Stolt

Part I

§ 1. Introduction

It is easy to solve the Diophantine equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with integral coefficients, in integers x and y when the equation represents an ellipse or a parabola in the (x, y)-plane. If the equation represents a hyperbola, the problem is much more difficult. In this case the problem may be reduced to the solution of the equation

$$u^2 - Dv^2 = \pm N, \tag{1}$$

where D and N are integers. We exclude the case of D being a perfect square, which is without interest. For solving an equation of this type one may use either the theory of quadratic forms or the theory of quadratic fields.

T. Nagell has shown[1] how it is possible to determine all the solutions of (1) independently of these theories.

Suppose that (1) is solvable, and let u and v be two integers satisfying (1). Then $u + v\sqrt{D}$ is called a *solution* of (1). If $x + y\sqrt{D}$ is a solution of the Diophantine equation

$$x^2 - Dy^2 = 1, \tag{2}$$

the number

$$(u + v\sqrt{D})(x + y\sqrt{D}) = (u_1 + v_1\sqrt{D})$$

is also a solution of (1). This solution is said to be *associated* with the solution $u + v\sqrt{D}$. The set of all solutions associated with each other forms a *class of solutions* of (1).

A necessary and sufficient condition for the two solutions $u + v\sqrt{D}$ and $u' + v'\sqrt{D}$ to belong to the same class is that the two expressions

$$\frac{uu' - vv'D}{N}, \quad \frac{vu' - uv'}{N} \tag{3}$$

be integers.

[1] See [1], [2], [3], [4]. In the following we use the notions proposed by Nagell.