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On the highest prime-power which divides n!

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This paper deals with the following problem[1]: Let p be a given prime and consider the numbers $1 \cdot 2 \cdot 3 \ldots n = n!$ for $n = 1, 2, 3$ etc. Find the integral exponents m with the property that p^m cannot be the highest power of p dividing $n!$ for any n. We call these numbers m the exceptional exponents of p.

Put $n = \sum_{\nu=0}^{h} a_\nu p^\nu$ and $s = \sum_{\nu=0}^{h} a_\nu$, where $a_0, a_1, \ldots, a_h$ are integers such that $0 \leq a_\nu \leq p-1$. When $e(n)$ denotes the exponent of the highest power of p dividing $n!$, we have by Legendre's formula

$$e(n) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right] = \frac{n-s}{p-1}.$$

The smallest exceptional exponent is clearly $m = p$, for $e(p^2 - 1) = p - 1$ and $e(p^2) = p + 1$. As n increases, new numbers m will appear as often as n is a multiple of p^2.

$n = p^h$ gives $h - 1$ new numbers m. For simplicity we write e_h for $e(p^h)$.

Since $p^h = p \cdot p^{h-1}$, this gives the recursion formula

$$e_1 = 1, \quad e_h = p\, e_{h-1} + 1.$$

Thus

$$e_h = p^{h-1} + p^{h-2} + \cdots + 1 = \frac{p_h - 1}{p-1},$$

as can easily be shown by induction.

Hence

$$m = \frac{p^h - 1}{p-1} - \varrho = e_h - \varrho \qquad (\varrho = 1, 2, \ldots, h-1)$$

are the new exceptional exponents for $n = p^h$. Consider the general case

$$n = \sum_{\nu=0}^{h} a_\nu p^\nu, \quad (0 \leq a_\nu \leq p-1).$$

[1] Proposed by T. NAGELL in Problem 43, p. 123 in his "Elementär talteori", Uppsala 1950.