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On singular monotonic functions whose spectrum has a given Hausdorff dimension

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1. This paper deals exclusively with continuous monotonic functions which are singular and of the Cantor type, that is to say which are constant in each interval contiguous to a perfect set of measure zero. This perfect set will be called the spectrum of the function.

We shall first prove the following results:

Theorem I. *Given any number α, $0 < \alpha < 1$, and a positive ε, arbitrarily small, but fixed, there exists a perfect set E, with Hausdorff dimension α, and a non-decreasing function $F(x)$, singular, with spectrum E, such that the Fourier Stieltjes transform of dF belongs to L^q for every $q \geq \frac{2}{\alpha} + \varepsilon$.*

Theorem II. *Given any number α, $0 < \alpha < 1$, and a positive ε, arbitrarily small, but fixed, there exists a perfect set E, with Hausdorff dimension α, and a non-decreasing function $F(x)$, singular, with spectrum E, such that the Fourier-Stieltjes coefficients of dF are of order $1/n^{\frac{\alpha}{2}-\varepsilon}$.*

Remarks.

1). Theorem I could be deduced from Theorem II, but since the method of the proof is the same, we prove both theorems.

2). Theorem I has been proved in an earlier paper[1] for the case $\alpha = 1$ (the Lebesgue measure of the set being of course zero), even in the stronger form, that the Fourier Stieltjes transform of the singular function belongs to L^q for every $q > 2$. The argument is the same as in the present paper, although much simpler.

We next prove:

Theorem III. *No singular function (except constant) exists having as spectrum a perfect set of Hausdorff dimension $\alpha > 0$, and whose Fourier-Stieltjes transform belongs to L^q for some $q < \frac{2}{\alpha}$.*

[1] R. SALEM. On sets of multiplicity for trigonometrical series. American Journal of Mathematics, Vol. 64 (1942), pp. 531–538.