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On the growth of minimal positive harmonic functions in a plane region

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1. We consider a plane, open, connected region D, the case of infinite connectivity being most interesting. This paper has its starting-point in an attempt to characterize boundary points in a way more suited for purposes of function theory than is possible with the aid of the purely geometric concept *accessible*. Let us fix an arbitrary accessible boundary point P of D, defined by a system of equivalent curves from some inner point to P. Two continuous curves Γ_1 and Γ_2 in our region, ending at P, are said to be equivalent, if there exist curves in D situated arbitrarily close to P which join a point of Γ_1 with a point of Γ_2.

We now suppose that there exist in D *positive harmonic functions, tending to zero in the vicinity of every boundary point except P;* we denote the class of these functions by U_P. That U_P is non-void evidently implies a certain regularity of the region D; for simplicity we have chosen a definition of U_P somewhat more restrictive than is necessary for the following.

2. One may ask how to generate functions of U_P. A procedure, near at hand, is to start from the (generalized) Green's function $G(M_1, M_2)$ for D and a sequence of inner points $P_0, P_1, \ldots, P_n, \ldots$, converging to P, then form the quotients

$$q_n(M) = \frac{G(M, P_n)}{G(P_0, P_n)}, \qquad n = 1, 2, \ldots, \tag{1}$$

and finally take limit functions of the family $\{q_n(M)\}_1^\infty$, normal in every closed part of D. Every member of U_P can be linearly expressed by such limit functions (MARTIN [4]). On the other hand, as instances of irregularity, we can construct regions where boundary continua are so accumulated towards an accessible boundary point P that it holds true for every limit function of (1) that

(a) it tends to infinity in the vicinity of a whole boundary continuum, or that

(b) it tends to zero in the vicinity of every boundary point except $P' \neq P$.

3. We now turn to a closer examination of the class U_P. If all functions of the class are proportional, we define P as *harmonically simple*, otherwise *multiple*. We shall not examine here how to distinguish between these eventualities, a question which we have had occasion to investigate somewhat in another con-