ARKIV FÖR MATEMATIK Band 1 nr 24

Communicated 7 June 1950 by F. CARLSON

On the coefficients in the power series expansion of a rational function with an application on analytic continuation

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The coefficients of the power series

$$\frac{2}{2-x}=\frac{1}{1}+\frac{1}{2}x+\cdots+\frac{1}{2^n}x^n+\cdots=\sum_{n=0}^{\infty}\frac{\alpha_n}{\beta_n}x^n$$

have the property that the denominators β_n tend to infinity exponentially with the index n. Let the function $\frac{2}{2-x}$ be replaced by another *rational* function and the coefficients $\frac{\alpha_n}{\beta_n}$ $((\alpha_n, \beta_n) = 1)$* be altered accordingly. Might it then occur that simultaneously $\overline{\lim\limits_{n=\infty}}\, \beta_n = \infty$ and $\beta_n = O(n^k)$? We shall give an answer in the negative in proving the following theorem:

Let $r(x)$ be a rational function, which in the neighbourhood of $x = 0$ is represented by a power series with rational coefficients, whose reduced forms are $\frac{\alpha_n}{\beta_n}$,

$$r(x)=\frac{\alpha_0}{\beta_0}+\frac{\alpha_1}{\beta_1}x+\cdots+\frac{\alpha_n}{\beta_n}x^n+\cdots$$

$$\alpha_n,\ \beta_n \textit{ integers},\ (\alpha_n, \beta_n) = 1 \qquad n = 1, 2, 3, \ldots$$

$$\beta_n = 1, \textit{ when } \alpha_n = 0.$$

*Then, either the sequence $|\beta_n|$ $(n = 1, 2, 3, \ldots)$ is bounded or $\overline{\lim\limits_{n=\infty}} \sqrt[n]{|\beta_n|} > 1$.***

The proof is given in the sections 1 and 2 below. In section 3 there is an application on analytic continuation in connection with a paper[1] by Professor F. CARLSON. The results of this note were also suggested by him.

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* (a, b) means the highest common divisor of a and b.

** We put $\beta_n = 1$, when $\alpha_n = 0$, for the sake of brevity. In reality, only those values of n are considered for which $\alpha_n \neq 0$. A remark of this kind is relevant sometimes also in the sequel.

[1] CARLSON, Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Z., 9 (1921), p. 1–13.