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On a closure problem

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Let $f(x)$ be a measurable function defined on the real line and such that $|f|^p$ is summable for any $p \geq 1$. Under this condition we shall consider the closure properties of the set

$$f(x+t) \qquad (-\infty < t < \infty) \tag{1}$$

in the different spaces L^p for $p \geq 1$. By C_f^p we shall denote the linear closed subset of L^p spanned by (1) in the strong topology of this space.

According to a theorem of F. Riesz and Banach C_f^p is a proper subset of L^p if and only if there is a non-trivial solution $g \in L^q$ $(1/p + 1/q = 1)$ of the integral equation

$$0 = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi. \tag{2}$$

If this is the case for a certain $p > 1$ we get another non-trivial solution by setting

$$h(x) = \int_{x}^{x+1} g(x)\, dx,$$

which is bounded and therefore belongs to any space $L^{q'}$, $q' > q$. On applying the cited theorem once again we find that $C_f^p \neq L^p$ implies $C_f^{p'} \neq L^{p'}$ for $1 \leq p' < p$.

From this we conclude that there will exist in the general case a number $\gamma > 1$ such that the system (1) is closed on L^p for all $p > \gamma$ but not for any $p < \gamma$. If (1) is always, respectively never, closed on L^p $(p > 1)$, we obviously have to define γ as 1 or $+\infty$. This number γ shall be called the "closure exponent" of f, and our object is to study the relation between γ and the Hausdorff dimension α of the set E where the Fourier transform of f vanishes. According to two theorems of Wiener [5] we know that $\gamma = 1$ if E is empty, while $\gamma \leq 2$ if E is of vanishing linear measure. It is also known (Segal [4]) that this latter condition does not imply $\gamma < 2$ in the general case. We shall now prove the following