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An extremal property of the Riemann zeta-function

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Introduction

When it is known *a priori* that a function φ is the solution of a certain extremal problem, this very fact usually enables us to derive without difficulty the characteristic properties of the function. If, however, the same function is explicitly defined while the extremal problem is unknown, we find ourselves in a quite different situation where it may be extremely difficult to recognize these same properties of φ.

In the special case

$$\varphi(s) = \zeta(s) = \sum_1^\infty \frac{1}{n^s} \qquad (\sigma > 1)$$

we have an explicit definition of the Riemann zeta-function from which the "elementary" properties of ζ are easily derived, whereas this definition gives very poor information as regards the truth of the Riemann hypothesis and other "deeper" properties of ζ. It should also be recalled that all this properties which we believe to be true but cannot verify, indicate that ζ in a certain sense is of minimal order of magnitude.

It therefore seems worth while to ask whether ζ is the solution of an hitherto unknown extremal problem, for example, whether in a certain class C of functions, ζ minimizes an integral of the form[1]

$$\int_{-\infty}^{\infty} |\varphi(\tfrac{1}{2} + it)|^2 p(t)\, dt. \qquad (p > 0)$$

Suppose this is so, and suppose futhermore the class C has the following property: For any $\varphi \in C$ such that $\varphi(\alpha \pm i\beta) = \varphi(1 - \alpha \pm i\beta) = 0$, $\alpha < \frac{1}{2}$, this quadruple of zeros may be displaced in such a way that φ remains in C while its modulus decreses on $\sigma = \frac{1}{2}$. Then the Riemann hypothesis is obviously true.

However this may be, it certainly is an interesting problem to investigate the extremal properties of the zeta-function. The main purpose of this paper

[1] Except for the question of the existence of classes C and weight functions p of this kind, which will not be considered here, this paper is a summary of a lecture given at the Harvard Mathematical Colloquium in 1949.