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On the analytic continuation of Eulerian products

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1. Introduction and summary

1.1. Let $h(z)$ be an analytic function that is regular and takes the value 1 for $z = 0$ and has no limit-point of zeros or singularities in the region $|z| \leq 1$. Consider the formal Eulerian product

$$f(s) = \prod_p h(p^{-s}) \tag{1.1}$$

where p runs through all prime numbers, and

$$s = \sigma + i\tau$$

is a complex variable. We have, e. g.

$$h(z) = (1-z)^{-1} \qquad f(s) = \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1.2}$$

$$h(z) = (1-z) \qquad f(s) = \zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) \cdot n^{-s} \tag{1.3}$$

$$h(z) = \prod_{\nu=1}^{k} (1-z^{\nu})^{-\beta_\nu} \qquad f(s) = \prod_{\nu=1}^{k} \zeta(\nu s)^{\beta_\nu} \tag{1.4}$$

$$h(z) = e^z \qquad f(s) = e^{P(s)} \tag{1.5}$$

where

$$P(s) = \sum p^{-s} \tag{1.6}$$

p running through all primes.

The main purpose of this paper is to show[1]

Theorem I. *The imaginary axis is a natural boundary of $f(s)$, except for the case in which the functions $h(z)$ and $f(s)$ have the form* (1.4).

A wider class $\{h(z)\}$ is discussed in section 4.2., and in Part 5 the corresponding results are derived for functions of the form

$$\prod_p h(\chi(p) \cdot p^{-s}).$$

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