ARKIV FÖR MATEMATIK Band 1 nr 35

Communicated 24 January 1951 by H. Cramér and O. Klein

On empirical spectral analysis of stochastic processes

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1. Introduction

Two main cases of empirical determination of the spectrum of a stationary stochastic process will be treated in this paper.

To a given frequency, which all the time will be chosen as zero, there corresponds a discrete line in the spectrum. One wants to determine its mean amplitude when a realization of the process has been observed. Among all possible ways of estimating the mean amplitude there is one which gives maximum precision. If the rest of the spectrum is known *a priori* this estimate can be explicitly constructed. It is shown that this construction is related to the problem of prediction. For the construction of the asymptotically best estimate the knowledge of the rest of the spectrum is not necessary, at least not for purely non-deterministic processes.

The other case arises when the spectrum is absolutely continuous and one wants to estimate the spectral intensity or, what is equivalent, the covariance function. A class of estimates is given and studied in relation to periodogram analysis and to an estimate proposed by Bartlett [2]. A principle of uncertainty is stated.

In passing, some simple properties of linear processes are studied.

2. Estimation and prediction

2.1. Let $x(t)$, $-\infty < t < \infty$, be a real valued process with finite variance. The mean value m and the covariance function $\varrho(s, t)$ are defined as

$$\begin{cases} m = E\,x(t) \\ \varrho(s, t) = E\,[x(s) - m]\,[x(t) - m]. \end{cases}$$

The process is supposed to be continuous in the mean. Consider $x(t)$ in an interval $T = (a, b)$. Introduce the inner product

$$(y, z) = E\,y\,z$$

and the norm

$$\|y\| = \sqrt{(y, y)},$$

for any stochastic variables y and z with finite variance. The Hilbert space generated by $x(t)$ when t runs through T is denoted $L_2(T)$. When T is the whole real axis we shall write $L_2(T) = L_2$.