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Spectral synthesis of bounded functions[1]

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This paper is intended as a sequel to the elegant paper of GODEMENT [4][2] on harmonic analysis. The fundamental problem, first posed by BEURLING, is this: when is a bounded measurable function φ defined on a locally compact abelian group G the weak limit of linear combinations of characters belonging to the spectral set Λ_φ of φ? In the dual language of L^1, the problem is to determine when a closed ideal of the group algebra is the intersection of regular maximal ideals.

SCHWARTZ [8] has given an example showing that the spectral approximation is not possible for all functions in Euclidean space of three dimensions. On the other hand the approximation is known to be possible if G is the real line and if Λ_φ is assumed to have denumerable (or reducible) boundary. Proofs of this theorem have been published by DITKIN [2], SEGAL [9], and MANDELBROJT and AGMON [6]. By using Segal's method and the structure theory of groups, KAPLANSKY [5] extended the theorem to a wide class of groups; and actually to arbitrary G if Λ_φ contains only one point.

This theorem of KAPLANSKY states that if Λ_φ consists of a single point, then φ is itself a multiple of a character. A proof based on the theory of distributions was found independently by JEAN RISS [7]. Our first objective is to give a new proof of this theorem using only simple analysis on the group itself. This seems desirable for aesthetic reasons, but more important, the structure theory is evidently not always enough to extend results from the real line to arbitrary groups.

Second, we extend an unpublished proof of BEURLING to obtain for arbitrary groups the theorem quoted above for the real line, thus accomplishing what KAPLANSKY set out to do by structure theory.

Finally, we modify a theorem of von Neumann and Dixmier about operators on Hilbert space to show that if the spectral approximation of φ is possible in the weak topology, then it is also possible in certain stronger topologies.

§ 1. Introduction

We begin with some definitions and theorems from GODEMENT [4] which will be used without reference hereafter.

[1] The content of § 2 and part of Theorem 2 appeared in my thesis. I wish to thank Professor LYNN LOOMIS most cordially for his direction. The other parts have roots in conversations with Professor ARNE BEURLING.

[2] Numbers in brackets refer to references at the end of the paper.