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A remark on a theorem by Frostman

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The origin of this remark is a lecture held by Frostman in Helsinki 1957 [2].

Let us introduce some definitions and notations.

Let K be an arbitrary compact set in the euclidean space R^n and let α be a number such that $0<\alpha<n$. Put

$$\|\mu\|_\alpha^2=\int\int\frac{d\mu(x)\,d\mu(y)}{|x-y|^{n-\alpha}}$$

where μ is a distribution of mass in R^n and where x and y denote points in R^n, $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$.

A is the set of all positive distributions of unit mass on K, that is,

$$\mu\geqslant 0,\quad \mu(K)=1,\quad \mu(R^n-K)=0.$$

Let $C_\alpha(K)$ be the capacity of K of order α

$$C_\alpha(K)=\frac{1}{\inf\limits_{\mu\in A}\|\mu\|_\alpha}.$$

It is well known that if $C_\alpha(K)>0$ then there exists a uniquely determined distribution μ_α in A that satisfies

$$\|\mu_\alpha\|_\alpha=\inf_{\mu\in A}\|\mu\|_\alpha.$$

μ_α is called the equilibrium distribution of order α on K. Frostman [2] has set the problem whether these equilibrium distributions vary continuously with α or not. Or, if $\alpha\searrow\beta$ ($\searrow$ means "tends non-increasingly to"), is it then true that μ_α converges towards a uniquely determined limit? (Convergence here in the weak sense, that is, $\mu_\alpha\to\mu$ is equivalent to $\int f\,d\mu_\alpha\to\int f\,d\mu$ for all continuous functions f with compact supports.)

If $C_\beta(K)>0$, the answer is yes. The limit in this case is μ_β which is easy to prove [2]. On the other hand, if $C_\beta(K)=0$, $C_\alpha(K)>0$ for $\alpha>\beta$, then the problem is not solved but for special cases. Frostman treats such a special case in [2] namely the case that $\alpha\searrow 1$ and that K is a curve in the plane ($n=2$) which is rectifiable. He proves that in this case $\mu_\alpha\to\mu_0$ where μ_0 is the distribution in A for which the mass which is situated on an arc is proportional to the lenght of that arc.